Iranian Journal of Mathematical Sciences and Informatics
Vol. 19, No. 1 (2024), pp 193-209
DOI: 10.61186/ijmsi.19.1.193

Fourier Method For an Existence of Quasilinear Inverse Pseudo-Parabolic Equation

Irem Baglan[a], Fatma Kanca[b], Vishnu Narayan Mishra[c*]

[a]Department of Mathematics, Kocaeli University, Kocaeli 41380, Turkey
[b]Department of Computer Engineering, Fenerbahce University, Istanbul, Turkey
[c]Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India

E-mail: isakinc@kocaeli.edu.tr
E-mail: fatma.kanca@fbu.edu.tr
E-mail: vishnunarayanmishra@gmail.com

Abstract. In this work, the inverse quasi-linear pseudo-parabolic problem was investigated. We demonstrated the solution by the Fourier approximation. The inverse problem was first examined by linearizing and then used implicit finite difference schema for the numerical solution.

Keywords: Pseudo-parabolic problem, Fourier method, Nonlocal(periodic) boundary condition, Implicit method.

2000 Mathematics subject classification: 35K05, 35K29, 65M06.

1. Introduction

Recently, the study of parabolic inverse problems, i.e. the determination of some unknown function in a parabolic equation, has received much attention. The study of mathematical models for many important applications such as

*Corresponding Author

Received 6 November 2019; Accepted 22 April 2023

chemical diffusion, applications in heat conduction processes, population dynamics, thermoelasticity, medical science, electrochemistry, engineering, wide scope, chemical engineering The inverse problem of determining unknown coefficient in a quasi-linear parabolic equation has generated an increasing amount of interest from engineers and scientist. [2, 14, 7, 3, 13, 1]. The Fourier Transform is used in fields such as analysis, sound engineering, and processing. The Fourier Transform is used for a descriptive system for time. This is the routing of the signals.Pseudo-parabolic equations describe a variety of important physical processes, such as the seepage of homogeneous fluids through a fissured rock of nonlinear pseudo-parabolic equations [16, 15, 12]. However, periodic boundary and integral boundary conditions and inverse coefficient problems have not been investigated by many researchers due to the difficulties of these conditions. Such conditions are used in heat transfer, in life sciences, lunar theory. Boundary value problem for quasilinear parabolic equations with periodic boundary conditions was handled [8, 9, 1, 10, 11]. Baglan I., Kanca F. study quasilinear parabolic equations with periodic boundary conditions by integral boundary condition [9]. They also study Euler Bernoulli equations with periodic boundary conditions by integral boundary condition [10]. They handle two dimensions problems with periodic boundary conditions by integral boundary condition [11]. The Fourier method is one of the most suitable methods used for this type of problems by theorically. The Fourier method has applications in many fields, for example, generalized orthogonal polynomials theory, Laguerre polynomials, Bessel Functions [5, 4, 6].

We will use Fourier method and finite-difference method for the model of partial differential equation (1.1)-(1.4).

$$w_t - w_{xx} - \varepsilon w_{xxt} = r(t)h(x,t,w),\ (x,t) \in\ \Gamma, \tag{1.1}$$

$$w(x,0) = \vartheta(x), \quad x \in [0,\pi], \tag{1.2}$$

$$w(0,t) = w(\pi,t),\ w_x(0,t) = w_x(\pi,t),\ 0 \le t \le T, \tag{1.3}$$

$$w_{xx}(\pi,t) = g(t), 0 \le t \le T. \tag{1.4}$$

The condition (1.2) is called initial condition, conditions (1.3) are called periodic boundary conditions.Finding the pairs $\{w(x,t), r(t)\}$ is called the inverse problem.

Here $\Gamma := \{0 < x < \pi,\ 0 < t < T\}$. The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,\pi]$ and $\overline{\Gamma} \times (-\infty,\infty)$, respectively.

The existence, uniqueness and convergence of the weak generalized solution the direct problem (1.1)-(1.4) are considered in [8]. The numerical solution of this parabolic problem is considered in [1].

In this paper, we prove existence, uniqueness of the solution and we will develop the implicit finite-difference scheme for solving the pseudo-parabolic problem inverse problem with periodic boundary conditions.There are several methods aimed at the numerical approximation of parabolic inverse problems, for example implicit finite-difference schemes for the solution of parabolic inverse problem.

Definition 1.1. The pair $\{r(t), u(x,t)\}$ from the class $C[0,T] \times (C^{2,1}(\Gamma) \cap C^{1,0}(\overline{\Gamma}))$ for which conditions (1.1)-(1.4) are satisfied is called the classical solution of the inverse problem (1.1)-(1.4).

The paper is organized as follows. In Section 2, the existence and the uniqueness of the solution of the problem are proved using the Fourier method and iteration method. In Section 3, stability of method for the solution is shown. In Section 4, the numerical procedure for the solution of the problem is given.

2. Solution of this problem

First of all, Fourier method will be used to solve this problem. Then, the uniqueness and continuous dependency of the creation process will be demonstrated by creating iterations with the Picard approximation method.The main result on the existence and the uniqueness of the solution of the inverse problem (1.1)-(1.4) is presented by Picard approximation method.In order to solve this problem, it is necessary to satisfy the provisions of Picard's theorem.In order to prove the theory, we must define certain conditions.

We accept the following assumptions:

(K1) $g(t) \in C^1[0,T]$.

(K2) $\vartheta(x) \in C^2[0,\pi]$, $\vartheta(0) = \vartheta(\pi)$, $\vartheta'(0) = \vartheta'(\pi)$,

(K3)

(1)

$$\left| \frac{\partial^{(n)} h(x,t,w)}{\partial x^n} - \frac{\partial^{(n)} h(x,t,\widetilde{w})}{\partial x^n} \right| \leq l(x,t) \left| w - \widetilde{w} \right| \ , n = 0,1,2$$

where $l(x,t) \in L_2(\Gamma)$, $l(x,t) \geq 0$,

(2) $h(x,t,w) \in C^2[0,\pi]$, $t \in [0,T]$,

(3) $h(x,t,w)|_{x=0} = h(x,t,w)|_{x=\pi}$, $h_x(0,t,w)|_{x=0} = h_x(\pi,t,w)|_{x=\pi}$,

(4) $r(t) \in C[0,T]$:

By the Fourier method, we have

$$w(x,t) = \frac{w_0(t)}{2} + \sum_{k=1}^{\infty} \left[w_{ck}(t) \cos 2kx + w_{sk}(t) \sin 2kx \right].$$

If we integrate the previous equation,we integrate from 0 to π,we find $w_0(t)$, we integrate from 0 to π and multiply by the $\cos 2k\alpha$, we find $w_{ck}(t)$, we integrate from 0 to π and multiply by the $\sin 2k\alpha$, we find $w_{sk}(t)$, respectively.

$$\begin{aligned} w_0(t) &= \vartheta_0 \\ &+ \frac{2}{\pi} \int_0^t \int_0^\pi r(\beta) h(\alpha, \beta, w)\, d\alpha d\beta, \\ w_{ck}(t) &= \vartheta_{ck}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} \\ &+ \frac{2}{\pi(1+\varepsilon(2k)^2} \int_0^t \int_0^\pi r(\beta) h(\alpha, \beta, w) \cos 2k\alpha\, e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\, d\alpha d\beta, \\ w_{sk}(t) &= \vartheta_{sk}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} \\ &+ \frac{2}{\pi(1+\varepsilon(2k)^2} \int_0^t \int_0^\pi r(\beta) h(\alpha, \beta, w) \sin 2k\alpha\, e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\, d\alpha d\beta, \end{aligned}$$

where $w_0(t), w_{ck}(t), w_{sk}(t)$ are Fourier coefficients. If we substitute these values in the equation,we have

$$\begin{aligned} w(x,t) &= \vartheta_0 + \int_0^t r(\beta) h_0(\beta, w)\, d\beta \\ &+ \sum_{k=1}^{\infty} \cos 2kx \left[\vartheta_{ck}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} \right] \\ &+ \sum_{k=1}^{\infty} \frac{1}{1+\varepsilon(2k)^2} \int_0^t r(\beta) h_{ck}(\beta, w)\, e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\, d\beta \\ &+ \sum_{k=1}^{\infty} \sin 2kx \left[\vartheta_{sk}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} \right] \\ &+ \sum_{k=1}^{\infty} \frac{1}{1+\varepsilon(2k)^2} \int_0^t r(\beta) h_{sk}(\beta, w)\, e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\, d\beta. \end{aligned} \tag{2.1}$$

Using (K1)-(K3), differentiating (1.4), we get

$$w_{xxt} = g^{'}(t), 0 \le t \le T. \tag{2.2}$$

From (2.1) and (2.2)

$$\begin{aligned} r(t) &= \frac{g(t) + \varepsilon g^{'}(t)}{h_0(t) + \sum_{k=1}^{\infty} \frac{h_{ck}(\tau,w)}{1+\varepsilon(2k)^2} - h(\pi,t,w)} \\ & \frac{\sum_{k=1}^{\infty} \frac{-(2k)^2 t}{1+\varepsilon(2k)^2} \left(\varphi_{ck}\, e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} + \frac{1}{1+\varepsilon(2k)^2} \int_0^t r(\beta) h_{ck}(\beta,w)\, e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\, d\beta \right)}{h_0(t) + \sum_{k=1}^{\infty} \frac{h_{ck}(\tau,w)}{1+\varepsilon(2k)^2} - h(\pi,t,w)} \end{aligned}$$

$$\begin{aligned} h(\pi,t,w) &= \frac{h_0(t)}{2} + \sum_{k=1}^{\infty} [h_{ck}(t) \cos 2k\pi + h_{sk}(t) \sin 2k\pi] \\ &= \frac{h_0(t)}{2} + \sum_{k=1}^{\infty} h_{ck}(t) \end{aligned}$$

$$r(t) = \frac{g(t)+\varepsilon g'(t)}{\frac{h_0(t)}{2}-\sum\limits_{k=1}^{\infty}\frac{\varepsilon(2k)^2 h_{ck}(t)}{1+\varepsilon(2k)^2}} + \frac{\sum\limits_{k=1}^{\infty}\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}\left(\varphi_{ck}\ e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}}+\frac{1}{1+\varepsilon(2k)^2}\int\limits_0^t r(\beta)h_{ck}(\beta,w)\ e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}d\beta\right)}{\frac{h_0(t)}{2}-\sum\limits_{k=1}^{\infty}\frac{\varepsilon(2k)^2 h_{ck}(t)}{1+\varepsilon(2k)^2}} \tag{2.3}$$

Definition 2.1. $\{w(t)\}=\{w_0(t),w_{ck}(t),w_{sk}(t),k=1,...,n\}$ satisfy such that

$\max\limits_{0\le t\le T}\frac{|w_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|w_{ck}(t)|+\max\limits_{0\le t\le T}|w_{sk}(t)|\right)<\infty$ by **B**.

$\|w(t)\|_B=\max\limits_{0\le t\le T}\frac{|w_0(t)|}{2}+\sum\limits_{k=1}^{\infty}\left(\max\limits_{0\le t\le T}|w_{ck}(t)|+\max\limits_{0\le t\le T}|w_{sk}(t)|\right)$ be norm of B.

Theorem 2.2. *If (K1)-(K3) be ensured. Then there is a solution to the problem (1.1)-(1.4).*

Proof. Let iterations to equation (2.1)

$$\begin{aligned}
w_0^{(N+1)}(t) &= w_0^{(0)}(t)+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi r^{(N)}(\beta)\ h\left(\alpha,\beta,w^{(N)}(\alpha,\beta)\right)d\alpha d\beta, \qquad (2.4)\\
w_{ck}^{(N+1)}(t) &= w_{ck}^{(0)}(t)\\
&+\frac{2}{\pi(1+\varepsilon(2k)^2)}\int\limits_0^t\int\limits_0^\pi r^{(N)}(\beta)\ h\left(\alpha,\beta,w^{(N)}(\alpha,\beta)\right)\cos 2k\alpha\ e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\ d\alpha d\beta,\\
w_{sk}^{(N+1)}(t) &= w_{sk}^{(0)}(t)\\
&+\frac{2}{\pi(1+\varepsilon(2k)^2)}\int\limits_0^t\int\limits_0^\pi r^{(N)}(\beta)\ h\left(\alpha,\beta,w^{(N)}(\alpha,\beta)\right)\sin 2k\alpha\ e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\ d\alpha d\beta,
\end{aligned}$$

where $w_0^{(0)}(t)=\vartheta_0, w_{ck}^{(0)}(t)=\vartheta_{ck}e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}}, w_{sk}^{(0)}(t)=\vartheta_{sk}e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}}$.

Adding and subtracting
$\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi r^{(0)}(\beta)\ h(\alpha,\beta,0)\,d\alpha d\beta,\ \frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi r^{(0)}(\beta)\ h(\alpha,\beta,0)\cos 2k\alpha e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}d\alpha d\beta,$
$\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi r^{(0)}(\beta)\ h(\alpha,\beta,0)\sin 2k\alpha e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}d\alpha d\beta$

to both sides of the last equation and using Cauchy inequality, we get

$$
\begin{aligned}
\left|w^{(1)}(t)\right| \leq\ & \frac{|\varphi_0|}{2} + \sum_{k=1}^{\infty}\left(|\varphi_{ck}| + |\varphi_{sk}|\right) \\
& + \left(\int_0^t d\beta\right)^{\frac{1}{2}} \left(\int_0^t \left(\frac{2}{\pi}\int_0^\pi r^{(0)}(\beta)\ \left(h(\alpha,\beta,w^{(*)})\right) d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \left(\int_0^t d\beta\right)^{\frac{1}{2}} \left(\int_0^t \left(\frac{2}{\pi}\int_0^\pi r^{(0)}(\beta)\ h\left(\alpha,\beta,0\right) d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \sum_{k=1}^{\infty}\left(\int_0^t e^{\frac{-2(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}} d\beta\right)^{\frac{1}{2}} \\
& \times \left(\int_0^t \left(\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^\pi r^{(0)}(\beta)\ \left(h(\alpha,\beta,w^{(*)})\right)\cos 2k\alpha d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \sum_{k=1}^{\infty}\left(\int_0^t e^{\frac{-2(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}} d\beta\right)^{\frac{1}{2}} \\
& \times \left(\int_0^t \left(\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^\pi r^{(0)}(\beta)\ h\left(\alpha,\beta,0\right)\cos 2k\alpha d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \sum_{k=1}^{\infty}\left(\int_0^t e^{\frac{-2(2k)^2(t-\beta)}{1+\varepsilon(2k)x^2}} d\beta\right)^{\frac{1}{2}} \\
& \times \left(\int_0^t \left(\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^\pi r^{(0)}(\beta)\left(h(\alpha,\beta,w^{(*)})\right)\sin 2k\alpha d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \sum_{k=1}^{\infty}\left(\int_0^t e^{\frac{-2(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}} d\beta\right)^{\frac{1}{2}} \\
& \times \left(\int_0^t \left(\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^\pi r^{(0)}(\beta)h\left(\alpha,\beta,0\right)\sin 2k\alpha d\alpha\right)^2 d\beta\right)^{\frac{1}{2}}
\end{aligned}
$$

where $h\left(\alpha,\beta,w^{(0)}\right) - h\left(\alpha,\beta,0\right) = h(\alpha,\beta,w^{(*)})$

$$\begin{aligned}
\left|w^{(1)}(t)\right| \leq\ & \frac{|\varphi_0|}{2} + \sum_{k=1}^{\infty}(|\varphi_{ck}| + |\varphi_{sk}|) \\
& + \frac{\sqrt{T}}{\sqrt{\pi}}\left(\int_0^t \left(\int_0^\pi r^{(0)}(\beta)\ d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \frac{\sqrt{T}}{\sqrt{\pi}}\left(\int_0^t \left(\int_0^\pi r^{(0)}(\beta)\ h(\alpha,\beta,0)\, d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \frac{\sqrt{2\pi}}{4\sqrt{3}}\sum_{k=1}^{\infty}\left(\int_0^t \left(\int_0^\pi r^{(0)}(\beta)\ \left(h(\alpha,\beta,w^{(*)})\right)\cos 2k\alpha d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \frac{\sqrt{2\pi}}{4\sqrt{3}}\sum_{k=1}^{\infty}\left(\int_0^t \left(\int_0^\pi r^{(0)}(\beta)\ h(\alpha,\beta,0)\cos 2k\alpha d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \frac{\sqrt{2\pi}}{4\sqrt{3}}\sum_{k=1}^{\infty}\left(\int_0^t \left(\int_0^\pi r^{(0)}(\beta)\ \left(h(\alpha,\beta,w^{(*)})\right)\sin 2k\alpha d\alpha\right)^2 d\beta\right)^{\frac{1}{2}} \\
& + \frac{\sqrt{2\pi}}{4\sqrt{3}}\sum_{k=1}^{\infty}\left(\int_0^t \left(\int_0^\pi r^{(0)}(\beta)\ h(\alpha,\beta,0)\sin 2k\alpha d\alpha\right)^2 d\beta\right)^{\frac{1}{2}}
\end{aligned}$$

Applying Hölder, Bessel inequality and using Lipschitzs condition and taking maximum

$$\begin{aligned}
\left\|w^{(1)}(t)\right\|_{\mathbf{B}} =\ & \max_{0\leq t\leq T}\frac{\left|w_0^{(1)}(t)\right|}{2} \\
& + \sum_{k=1}^{\infty}\left(\max_{0\leq t\leq T}\left|w_{ck}^{(1)}(t)\right| + \max_{0\leq t\leq T}\left|w_{sk}^{(1)}(t)\right|\right) \\
\leq\ & \frac{|\varphi_0|}{2} \\
& + \sum_{k=1}^{\infty}(|\varphi_{ck}| + |\varphi_{sk}|) \\
& + \left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\|l(x,t)\|_{L_2(\Gamma)}\left\|w^{(0)}(t)\right\|_B\left\|r^{(0)}(\tau)\right\|_{C[0,T]} \\
& + \left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left\|r^{(0)}(\tau)\right\|_{C[0,T]}\|h(x,t,0)\|_{L_2(\Gamma)}.
\end{aligned}$$

From the condition of the theorem we have $w^{(0)}(t) \in \mathbf{B}$. We prove that the other approximations in the sequence satisfy this condition, then $w^{(1)}(t) \in \mathbf{B}$.

For N,

$$\begin{aligned}\left\|w^{(N+1)}(t)\right\|_B &= \max_{0\le t\le T}\left|w_0^{(N)}(t)\right| \\ &+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|w_{ck}^{(N)}(t)\right|+\max_{0\le t\le T}\left|w_{sk}^{(N)}(t)\right|\right)\\ &\le |\vartheta_0| \\ &+\sum_{k=1}^{\infty}\left(|\vartheta_{ck}|+|\vartheta_{sk}|\right)\\ &+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\|l(x,t)\|_{L_2(\Gamma)}\left\|w^{(N)}(t)\right\|_B\left\|r^{(N)}(\tau)\right\|_{C[0,T]}\\ &+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left\|r^{(N)}(\tau)\right\|_{C[0,T]}\|h(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

we obtain $w^{(N+1)}(t)\in\mathbf{B}$.

$$\begin{aligned}r^{(1)}(t) &= \frac{g(t)+\varepsilon g'(t)}{\frac{h_0(t)}{2}-\sum\limits_{k=1}^{\infty}\frac{\varepsilon(2k)^2h_{ck}(t)}{1+\varepsilon(2k)^2}}\\ &+\frac{\sum\limits_{k=1}^{\infty}\frac{-(2k)^2t}{1+\varepsilon(2k)^2}\left(\varphi_{ck}\,e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}}+\frac{1}{1+\varepsilon(2k)^2}\int_0^t r^{(0)}(\beta)h_{ck}(\beta,w^{(0)})\,e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}d\beta\right)}{\frac{h_0(t)}{2}-\sum\limits_{k=1}^{\infty}\frac{\varepsilon(2k)^2h_{ck}(t)}{1+\varepsilon(2k)^2}}\end{aligned}$$

Applying Cauchy,Hölder inequality

$$\begin{aligned}\left|r^{(1)}(t)\right| &\le \frac{|g(t)|}{M_0}+\varepsilon\frac{\left|g'(t)\right|}{M_0}\\ &+\frac{\sum\limits_{k=1}^{\infty}\frac{-(2k)^2t}{\left(1+\varepsilon(2k)^2\right)(2k)^2}\left(\left|\varphi''_{ck}\right|+\frac{1}{1+\varepsilon(2k)^2}\int_0^t\int_0^\pi r^{(0)}(\beta)\left(h(\alpha,\beta,w^{(*)})\right)e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}d\beta\right)}{M_0}\\ &+\frac{\sum\limits_{k=1}^{\infty}\frac{-(2k)^2t}{\left(1+\varepsilon(2k)^2\right)(2k)^2}\left(\left|\varphi''_{ck}\right|+\frac{1}{1+\varepsilon(2k)^2}\int_0^t\int_0^\pi r^{(0)}(\beta)h(\alpha,\beta,0)\,e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}d\beta\right)}{M_0}\end{aligned}$$

Applying Hölder, Bessel inequality and using Lipschitzs condition and taking maximum, we have

$$\begin{aligned}\left\|r^{(1)}(t)\right\|_{C[0,T]} &\le \frac{|g(t)|}{M_0}+\varepsilon\frac{\left|g'(t)\right|}{M_0}\\ &+\frac{\pi}{M_0\sqrt{6}}\sum_{k=1}^{\infty}\left|\varphi''_{ck}\right|+\frac{\sqrt{3}}{3M_0}\left\|r^{(0)}(t)\right\|_{C[0,T]}\left\|w^{(0)}(t)\right\|_B\\ &+\frac{\sqrt{3}}{3M_0}\left\|r^{(0)}(t)\right\|_{C[0,T]}\|h(x,t,0)\|_{L_2(\Gamma)}\end{aligned}$$

where

$$\frac{h_0(t)}{2}+\sum_{k=1}^{\infty}\frac{\varepsilon(2k)^2h_{ck}(t)}{1+\varepsilon(2k)^2}\ge M_0.$$

For N,

$$\begin{aligned}\left\|r^{(N+1)}(t)\right\|_{C[0,T]} &\leq \frac{|g(t)|}{M_0}+\varepsilon\frac{\left|g^{'}(t)\right|}{M_0}\\&+\frac{\pi}{M_0\sqrt{6}}\sum_{k=1}^{\infty}\left|\varphi_{ck}^{''}\right|+\frac{\sqrt{3}}{3M_0}\left\|r^{(N)}(t)\right\|_{C[0,T]}\left\|w^{(N)}(t)\right\|_{B}\\&+\frac{\sqrt{3}}{3M_0}\left\|r^{(N)}(t)\right\|_{C[0,T]}\|h(x,t,0)\|_{L_2(\Gamma)}\end{aligned}$$

$w^{(N+1)}(t)\in\mathbf{B}$.

We show $w^{(N+1)}, r^{(N+1)}$ are converged for $N\to\infty$,

$$\begin{aligned}w^{(1)}(t)-w^{(0)}(t) &= \frac{(w_0^{(1)}(t)-w_0^{(0)}(t))}{2}\\&+\sum_{k=1}^{\infty}[(w_{ck}^{(1)}(t)-w_{ck}^{(0)}(t))+(w_{sk}^{(1)}(t)-w_{sk}^{(0)}(t))]\\&=\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^{\pi}r^{(0)}(\beta)\left[h(\alpha,\beta,w^{(*)})\right]d\alpha d\beta\right)\\&+\sum_{k=1}^{\infty}\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^{\pi}\left[r^{(0)}(\beta)h(\alpha,\beta,w^{(*)})\right]e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\cos 2\pi k\alpha d\alpha d\beta\\&+\sum_{k=1}^{\infty}\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^{\pi}\left[r^{(0)}(\beta)h(\alpha,\beta,w^{(*)})\right]e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\sin 2\pi k\alpha d\alpha d\beta\\&+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^{\pi}r^{(0)}(\beta)h(\alpha,\beta,0)d\alpha d\beta\right)\\&+\sum_{k=1}^{\infty}\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^{\pi}r^{(0)}(\beta)h(\alpha,\beta,0)\,e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\cos 2\pi k\alpha d\alpha d\beta\\&+\sum_{k=1}^{\infty}\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^{\pi}r^{(0)}(\beta)h(\alpha,\beta,0)\,e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\sin 2\pi k\alpha d\alpha d\beta.\end{aligned}$$

After using Cauchy, Hölder, Bessel inequality and Lipschitzs condition, we find

$$\begin{aligned}\left\|w^{(1)}(t)-w^{(0)}(t)\right\|_{B} &\leq \left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\|l(x,t)\|_{L_2(\Gamma)}\\&x\left\|r^{(0)}(\tau)\right\|_{C[0,T]}\left\|w^{(0)}(t)\right\|_{\mathbf{B}}\\&+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\\&x\left\|r^{(0)}(\tau)\right\|_{C[0,T]}\|h(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

$$\begin{aligned}A &= \left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left\|r^{(0)}(\tau)\right\|_{C[0,T]}\\&x(\|l(x,t)\|_{L_2(\Gamma)}\left\|w^{(0)}(t)\right\|_{\mathbf{B}}+\|h(x,t,0)\|_{L_2(\Gamma)}).\end{aligned}$$

$$\begin{aligned}\left\|r^{(1)}(t)-r^{(0)}(t)\right\|_{C[0,T]} &\leq \frac{1}{M_0}\left(\frac{\sqrt{3}M}{3-\sqrt{3}M^2}\right)\left\|r^{(1)}(t)\right\|_{C[0,T]} \\ &\quad x\left\|w^{(1)}(t)-w^{(0)}(t)\right\|_B \|l(x,t)\|_{L_2(\Gamma)}\end{aligned}$$

$$\begin{aligned}\left\|w^{(2)}(t)-w^{(1)}(t)\right\|_B &\leq \left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\|l(x,t)\|_{L_2(\Gamma)} \\ &\quad\times\left\|w^{(1)}(t)-w^{(0)}(t)\right\|_B\left\|r^{(1)}(\tau)\right\|_{C[0,T]} \\ &\quad+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)M\left\|r^{(1)}-r^{(0)}\right\|_{C[0,T]}.\end{aligned}$$

$$\begin{aligned}\left\|w^{(2)}(t)-w^{(1)}(t)\right\|_B &\leq \left\{\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left(1+\frac{\sqrt{3}M^2}{3-\sqrt{3}M^2}\right)\right\} \\ &\quad\times\frac{A}{M_0}\|l(x,t)\|_{L_2(\Gamma)}\left\|r^{(1)}(\tau)\right\|_{C[0,T]},\end{aligned}$$

For the step N :

$$\begin{aligned}\left\|r^{(N+1)}(t)-r^{(N)}(t)\right\|_{C[0,T]} &\leq \left(\frac{\sqrt{3}M}{3-\sqrt{3}M^2}\right)\frac{1}{M_0} \\ &\quad\times\left\|r^{(N+1)}(t)\right\|_{C[0,T]}\left\|w^{(N+1)}(t)-w^{(N)}(t)\right\|_B\|l(x,t)\|_{L_2(\Gamma)}\end{aligned}$$

$$\begin{aligned}\left\|w^{(N+1)}(t)-w^{(N)}(t)\right\|_B &\leq \frac{A}{\sqrt{N!}}\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left(1+\left(\frac{\sqrt{3}M^2}{3-\sqrt{3}M^2}\right)\frac{1}{M_0}\right)^N \\ &\quad\times\left\|r^{(1)}(\tau)\right\|_{C[0,T]}\left\|r^{(2)}(\tau)\right\|_{C[0,T]}\cdots\left\|r^{(N+1)}(\tau)\right\|_{C[0,T]}\|l(x,t)\|_{L_2(\Gamma)}^{N+1}.\end{aligned} \tag{2.5}$$

According to (2.5),we obtain
$w^{(N+1)}\to w^{(N)}$, $N\to\infty$, then $r^{(N+1)}\to r^{(N)}$, $N\to\infty$.
Let us show

$$\begin{aligned}\lim_{N\to\infty} w^{(N+1)}(t) &= w(t), \\ \lim_{N\to\infty} r^{(N+1)}(t) &= r(t).\end{aligned}$$

$$\begin{aligned}\left\|w-w^{(N+1)}\right\|_B &\leq \left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\|r(\tau)\|_{C[0,T]}\|l(x,t)\|_{L_2(\Gamma)}\left\|w(t)-w^{(N+1)}(t)\right\|_B \\ &\quad+\frac{A}{\sqrt{N!}}\left\{\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left(1+\frac{\sqrt{3}M^2}{3-\sqrt{3}M^2}\right)\right\}^N\|R(\tau)\|_{C[0,T]}\|l(x,t)\|_{L_2(\Gamma)} \\ &\quad+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)M\left\|r(\tau)-r^{(N)}(\tau)\right\|_{C[0,T]},\end{aligned} \tag{2.6}$$

where $\|r(\tau)\|_{C[0,T]}\left\|r^{(1)}(\tau)\right\|_{C[0,T]}\left\|r^{(2)}(\tau)\right\|_{C[0,T]}\cdots\left\|r^{(N+1)}(\tau)\right\|_{C[0,T]}=\|R(\tau)\|_{C[0,T]}$

$$\left\|r(\tau)-r^{(N+1)}(\tau)\right\|_{C[0,T]} \leq \left(\frac{\sqrt{3}M}{3-\sqrt{3}M^2}\right)\|r(\tau)\|_{C[0,T]} \times \|b(x,t)\|_{L_2(\Gamma)}\left\|w(t)-w^{(N+1)}(t)\right\|_{B}. \tag{2.7}$$

Let us consider (2.5) in (2.6) and apply Gronwall's inequality to (2.7), we obtain

$$\left\|w(t)-w^{(N+1)}(t)\right\|_{\mathbf{B}}^2 \leq 2\left[\frac{A}{\sqrt{N!}}\left(\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left(1+\frac{\sqrt{3}M^2}{3-\sqrt{3}M^2}\right)\right)^{N+1}\|R(\tau)\|_{C[0,T]}\|l(x,t)\|_{L_2(\Gamma)}\right]^2 \times \exp 2\left(1+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left(1+\frac{\sqrt{3}M^2}{3-\sqrt{3}M^2}\right)\right)^2\|l(x,t))\|_{L_2(\Gamma)}^2.$$

We have $w^{(N+1)} \to w$, we show (r,w) ,(q,v) are two solutions of (1.1)-(1.4). Applying same operations:

$$\|w(t)-v(t)\|_B \leq \left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)M\|r(t)-q(t)\|_{C[0,T]} + \left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left(\int_0^t\int_0^\pi r^2(\beta)l^2(\alpha,\beta)\left|w(\beta)-v(\beta)\right|^2 d\alpha d\beta\right)^{\frac{1}{2}},$$

$$\|r(t)-q(t)\|_{C[0,T]} \leq \left(\frac{\sqrt{3}M}{3-\sqrt{3}M^2}\right) \times\left(\int_0^t\int_0^\pi r^2(\beta)l^2(\alpha,\beta)\left|w(\beta)-v(\beta)\right|^2 d\alpha d\beta\right)^{\frac{1}{2}},$$

$$\|w(t)-v(t)\|_B \leq \left[\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left(1+\frac{\sqrt{3}M^2}{3-\sqrt{3}M^2}\right)\right] \times\left(\int_0^t\int_0^\pi r^2(\beta)l^2(\alpha,\beta)\left|w(\beta)-v(\beta)\right|^2 d\alpha d\beta\right)^{\frac{1}{2}}, \tag{2.8}$$

we get $w(t)=v(t)$. Hence $r(t)=q(t)$. □

3. Stability of the solution

In situations where a physical process is described (modeled) with PDEs, then it is desirable that any errors made in the measurement of either initial data or boundary data, do not influence the solution very much. In mathematical terms, this is known as continuous dependence of solution, on the data present in the problem. In fact, the following theorem asserts that solution (1.1)-(1.4) has continuous dependence on input data. Let's prove the following theorem.

Theorem 3.1. *If the conditions (C1)-(C3) are implemented then the solution of the problem depends continuously on ϑ, E.*

Proof. Let's make a slight difference to the initial data. Suppose $\Phi = \{\vartheta,\ g,\ h\}$ and $\overline{\Phi} = \left\{\overline{\vartheta},\ \overline{g},\ h\right\}$be two sets of the data, which satisfy the assumptions $(K1)-(K3)$. For $N_i,\ i=1,2$ such that

$$\|g\|_{C^1[0,T]} \leq N_1, \|\overline{g}\|_{C^1[0,T]} \leq N_1, \|\vartheta\|_{C^3[0,\pi]} \leq N_2, \left\|\overline{\vartheta}\right\|_{C^3[0,\pi]} \leq N_2.$$

Let us $\|\Phi\| = (\|g\|_{C^1[0,T]} + \|\vartheta\|_{C^3[0,\pi]} + \|h\|_{C^{3,0}(\overline{\Gamma})})$.

$$\begin{aligned} w-\overline{w} &= \frac{\left(\vartheta_0-\overline{\vartheta_0}\right)}{2} + \sum_{k=1}^{\infty}\left(\vartheta_{ck}-\overline{\vartheta_{ck}}\right)e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} + \sum_{k=1}^{\infty}\left(\vartheta_{sk}-\overline{\vartheta_{sk}}\right)e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} \\ &+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^\pi r(\beta)\left[h(\alpha,\beta,\overline{w}(*))\right]d\alpha d\beta\right) \\ &+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^\pi (r(\beta)-\overline{r}(\beta))\,h(\alpha,\beta,\overline{w}(\alpha,\beta))d\alpha d\beta\right) \\ &+\sum_{k=1}^{\infty}\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^\pi r(\beta)\left[h(\alpha,\beta,\overline{w}(*))\right]\cos 2k\alpha\; e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\,d\alpha d\beta \\ &+\sum_{k=1}^{\infty}\frac{2}{\pi(1+\varepsilon(2k)^2)} \\ &\times\int_0^t\int_0^\pi (r(\beta)-\overline{r}(\beta))\left[h(\alpha,\beta,\overline{w}(*))\right]\cos 2k\alpha\; e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\,d\alpha d\beta \\ &+\sum_{k=1}^{\infty}\frac{2}{\pi(1+\varepsilon(2k)^2)}\int_0^t\int_0^\pi r(\beta)\left[h(\alpha,\beta,\overline{w}(*))\right]\cos 2k\alpha\; e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\,d\alpha d\beta \\ &+\sum_{k=1}^{\infty}\frac{2}{\pi(1+\varepsilon(2k)^2)} \\ &\times\int_0^t\int_0^\pi (r(\beta)-\overline{r}(\beta))\left[h(\alpha,\beta,\overline{w}(*))\right]\cos 2k\alpha\; e^{\frac{-(2k)^2(t-\beta)}{1+\varepsilon(2k)^2}}\,d\alpha d\beta, \end{aligned}$$

where $h(\alpha,\beta,w(\alpha,\beta)) - h(\alpha,\beta,\overline{w}(\alpha,\beta) = h(\alpha,\beta,\overline{w}(*))$ and

$$r(\tau)-\overline{r}(\tau) = \left\{\begin{array}{c} \dfrac{g(t)+\varepsilon g'(t)+\sum\limits_{k=1}^{\infty}\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}\varphi_{ck}e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}}}{h_0(t,w)+\sum\limits_{k=1}^{\infty}\frac{\varepsilon(2k)^2 h_{ck}(t,w)}{1+\varepsilon(2k)^2}} \\ +\dfrac{\sum\limits_{k=1}^{\infty}\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}\left(\frac{1}{1+\varepsilon(2k)^2}\int_0^t r(\tau)h_{ck}(\tau,w)e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)}{h_0(t,w)+\sum\limits_{k=1}^{\infty}\frac{\varepsilon(2k)^2 h_{ck}(t,w)}{1+\varepsilon(2k)^2}} \\ -\dfrac{\overline{g(t)}+\varepsilon\overline{g'(t)}+\sum\limits_{k=1}^{\infty}\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}\overline{\varphi_{ck}}e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}}}{h_0(t,\overline{w})+\sum\limits_{k=1}^{\infty}\frac{\varepsilon(2k)^2 h_{ck}(t,\overline{u})}{1+\varepsilon(2k)^2}} \\ -\dfrac{\sum\limits_{k=1}^{\infty}\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}\left(\frac{1}{1+\varepsilon(2k)^2}\int_0^t \overline{r(\tau)}h_{ck}(\tau,\overline{w})e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}}d\tau\right)}{h_0(t,\overline{w})+\sum\limits_{k=1}^{\infty}\frac{\varepsilon(2k)^2 h_{ck}(t,\overline{u})}{1+\varepsilon(2k)^2}} \end{array}\right\}$$

After using Cauchy, Hölder, Bessel inequality and Lipschitzs condition, we find

$$\begin{aligned}\|r(\tau)-\overline{r}(\tau)\|_{C[0,T]} &\leq \frac{M_1}{M_2}\|g(t)-\overline{g}(t)\| \\ &+\varepsilon\frac{M_1}{M_2}\left\|g^{'}(t)-\overline{g^{'}(t)}\right\| \\ &+\frac{\pi M}{2\sqrt{6}}\sum_{k=1}^{\infty}\left|\vartheta_{ck}^{''}-\overline{\vartheta_{ck}^{''}}\right| \\ &+\frac{M_1}{M_2}\|w-\overline{w}\|\end{aligned}$$

$$\begin{aligned}\|w-\overline{w}\|_B &\leq M_3\left\|\Phi-\overline{\Phi}\right\| \\ &+M_4\left(\int_0^t\int_0^\pi r^2(\beta)l^2(\alpha,\beta)\|w(\beta)-\overline{w}(\beta)\|^2 d\alpha d\beta\right)^{\frac{1}{2}}\end{aligned} \tag{3.1}$$

$$\begin{aligned}\|w-\overline{w}\|_B^2 &\leq 2M_3^2\left\|\Phi-\overline{\Phi}\right\|^2 \\ &\times\exp\left(2M_4^2\int_0^t\int_0^\pi r^2(\beta)l^2(\alpha,\beta)d\alpha d\beta\right).\end{aligned} \tag{3.2}$$

According to the last equation,
for $\Phi\to\overline{\Phi}$ then $w\to\overline{w}$. Hence $r\to\overline{r}$. □

4. The Numerical Examination

We obtain the following problem after linearization of problem (1.1)-(1.4):

$$\frac{\partial w^{(n)}}{\partial t} = \frac{\partial^2 w^{(n)}}{\partial x^2}+\varepsilon\frac{\partial^3 w^{(n)}}{\partial x^2\partial t}+r(t)h(x,t,w^{(n-1)}), \qquad (x,t)\in D \tag{4.1}$$

$$w^{(n)}(0,t) = w^{(n)}(\pi,t),\quad w_x^{(n)}(0,t)=w_x^{(n)}(\pi,t)=0 \qquad t\in[0,T] \tag{4.2}$$

$$w_{xx}^{(n)}(\pi,t) = g(t),\ t\in[0,T] \tag{4.3}$$

$$w^{(n)}(x,0) = \vartheta(x)\ , \qquad x\in[0,\pi]\,. \tag{4.4}$$

Let $w^{(n)}(x,t)=v(x,t)$ and $h(x,t,w^{(n-1)})=\widetilde{h}(x,t)$. Linear problem for (4.1)-(4.4):

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}+\varepsilon\frac{\partial^3 v}{\partial x^2\partial t}+r(t)\widetilde{h}(x,t) \qquad (x,t)\in D \tag{4.5}$$

$$v(0,t) = v(\pi,t),v_x(0,t)=v_x(\pi,t), \qquad t\in[0,T] \tag{4.6}$$

$$v_{xx}(\pi,t) = g(t),\ t\in[0,T] \tag{4.7}$$

$$v(x,0) = \vartheta(x), \qquad x\in[0,\pi]\ . \tag{4.8}$$

The region $[0,\pi]$ and $[0,T]$ is reserved N_x and N_t of equal lengths $h=\frac{\pi}{N_x}$ and $\tau=\frac{T}{N_t}$, respectively. The implicit scheme of (4.5)-(4.8) :

$$\frac{1}{\tau}\left(v_i^{j+1}-v_i^j\right) = \frac{1}{2h^2}\left(v_{i-1}^j-2v_i^j+v_{i+1}^j\right) +\varepsilon\frac{1}{2h^2\tau}\left[(v_{i-1}^{j+1}-2v_i^{j+1}+v_{i+1}^{j+1})-(v_{i-1}^j-2v_i^j+v_{i+1}^j)\right]+r^j\widetilde{h}_i^j, \tag{4.9}$$

$$v_i^0=\phi_i, \tag{4.10}$$

$$v_0^j=v_{N_x+1}^j, \tag{4.11}$$

$$\frac{v_1^j+v_{N_x}^j}{2}=v_{N_x+1}^j, \tag{4.12}$$

$\left(v_i^j=v(x_i,t_j),\phi_i=\phi(x_i),\widetilde{h}_i^j=\widetilde{h}(x_i,t_j),x_i=ih,t_j=j\tau \text{ for } 1\le i\le N_x \text{ and } 0\le j\le N_t.\right)$

Integrate the equation (4.5) with respect to x from 0 to 1 and using (4.9) and (4.12), we obtain

$$r(t)=\frac{-g(t)-\varepsilon g'(t)+v_t(\pi,t)}{\widetilde{h}(\pi,t)dx}. \tag{4.13}$$

The finite-difference approximation of $r(t)$ is

$$r^j=\frac{-g(t)-\varepsilon\left(g^{j+1}-g^j\right)/\tau+\left(v_{N_x}^{j+1}-v_{N_x}^j\right)/\tau}{\widetilde{h}_{N_x}^j},$$

where $g^j=g(t_j)$, $j=0,1,...,N_t$. We mention that the integrals are numerically calculated using Simpson's rule of integration and also the first derivatives are calculated using central difference scheme $r^{j(s)}$, $v_i^{j(s)}$ are the values of r^j, v_i^j at the s-th iteration step, respectively. At each $(s+1)$-th iteration step, $r^{j+1(s+1)}$ is as follows

$$r^{j+1(s+1)}=\frac{-g(t)-\varepsilon\left(g^{j+2}-g^{j+1}\right)/\tau+\left(v_{N_x}^{j+1(s)}-v_{N_x}^{j(s)}\right)/\tau}{\widetilde{h}_{N_x}^j}.$$

The iteration to (4.9)-(4.12) we obtain

$$\begin{aligned}\frac{1}{\tau}\left(v_i^{j+1(s+1)}-v_i^{j+1(s)}\right) = {} & \frac{1}{h^2}\left(v_{i-1}^{j+1(s+1)}-2v_i^{j+1(s+1)}+v_{i+1}^{j+1(s+1)}\right)\\ &+\varepsilon\frac{1}{2h^2\tau}\left(v_{i-1}^{j+1(s+1)}-2v_i^{j+1(s+1)}+v_{i+1}^{j+1(s+1)}\right)\\ &-\varepsilon\frac{1}{2h^2\tau}\left(v_{i-1}^{j+1(s)}-2v_i^{j+1(s)}+v_{i+1}^{j+1(s)}\right)\\ &+r^{j+1(s+1)}\widetilde{h}_i^{j+1},\end{aligned} \tag{4.14}$$

$$v_0^{j(s)}=v_{N_x+1}^{j(s)}, \tag{4.15}$$

$$\frac{v_1^{j(s)}+v_{N_x}^{j(s)}}{2}=v_{N_x+1}^{j(s)}, \tag{4.16}$$

$v_i^{j+1(s+1)}$ is found.

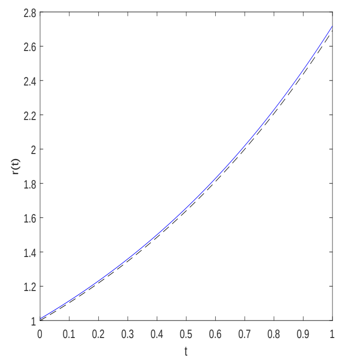

FIGURE 1. Exact and approximate r(t) when T=1.

The numerical method suggested here is the implicit Finite-difference method which is second order accurate in the spatial grid size and first-order in the time grid size. Also, the Crank–Nicolson scheme, which is absolutely stable and has a second-order accuracy in both the spatial and time grid sizes can be used. The explicit finite-difference schemes which are easy to use, have restriction on stability. The CPU time in the Crank–Nicolson scheme is longer than the implicit finite-difference method.

In order to illustrate the behavior of our numerical method, an example is considered.

EXAMPLE 4.1. Let the give functions are

$$\begin{aligned} h(x,t,w) &= (1+4\cos 2x-4\sin^2 x+4\varepsilon^2\cos 2x-4\varepsilon^2\sin^2 x)w\exp(-t), \\ \vartheta(x) &= \exp(\cos 2x),\ g(t)=-4\exp(\varepsilon t+1),\quad x\in[0,\pi]\,,\ t\in[0,T]\,. \end{aligned}$$

The analytical solution is

$$\{r(t),\ w(x,t)\}=\{\exp(t)\,,\exp\ (\varepsilon t+\cos(2x))\}\,.$$

Here $h=0.0393$, $\tau=0.01$.

Note that the convergence criterion for r(t) was
$\left|r^{k+1(s+1)}-r^{k+1(s)}\right|\leq h/200$.

This programming has been performed using the MATLAB R2016 version of the computational program and CPU time is found using the tic–toc command in MATLAB. The CPU time is 14 s.

For $T=1$, comparisons between the accurate solution and the numerical solution are given below in Figures 1 and 2 when $\varepsilon=0.1$.

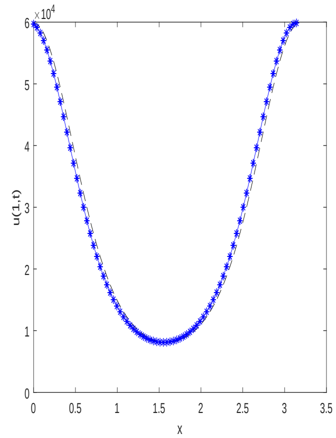

Figure 2. Exact and approximate solutions of u(x,t) at the T=1.

5. Conclusions

The problem of inverse of the temperature distribution in the pseudo- quasilinear parabolic equation by nonlocal conditions are taken . This inverse problem has been examined from both theoretically and numerically. In the theoretical part, the existence of the problem and the stability of the problem were examined. The finite difference method in the numerical part is preferred. Periodic conditions are used in this article.The Fourier and finite difference method are crucial of inverse problems for pseudo- quasilinear parabolic equations. This provides the insight into the modeling of problems with periodic boundary conditions. Periodic boundary conditions are more difficult than local boundary conditions for the inverse coefficient problems This work advances our understanding of the use of the Fourier method of separation of variables and the finite-difference methods in the investigation of inverse problems for pseduo-quasilinear parabolic equations. In future various theorical and numerical methods can be used for this problem. Also the authors plan to consider various linear and non-linear inverse problems with different boundary conditions in future studies, since the method discussed has a wide range of applications.

6. Acknowledgments

The authors would like to thank to the reviewers for their valuable comments.

References

1. I. Baglan, F. Kanca, The Continuous Dependence and Numerical Approximation of the Solution of the Quasilinear Pseudo-Parabolic Problem with Periodic Boundary Condition, *Journal of Advances in Mathematics*, **5**(3), (2014),132-143.
2. J. R. Cannon , Y. Lin, Determination of Parameter p(t) in Hölder Classes for Some Semilinear Parabolic Equations, *Inverse Problems*, **4**, (1988), 595-606.
3. Y. Cao, J. Yin, C. Wang, Cauchy Problems of Semilinear Pseudo-Parabolic Equations, *Journal of Differential Equations*, **246**(12), (2009), 4568-4590.
4. C. Cesarano, G. M. Cennamo, L. Pilacidil, Humbert Polynomials and Functions in Terms of Hermite Polynomials Towards Applications to Wave Propagation, *Wseas Transactions on Mathematics*, **13**, (2014), 595-60.
5. G. Dattoli, S. Lorenzutta, P. E. Ricci, C. Cesarano, On a Family of Hybrid Polynomials, *Integral Transforms and Special Functions*, **15**(6), (2004), 485-490.
6. G. Dattoli, P. E. Ricci, C. Cesarano, The Lagrange Polynomials the Associated Generalizations and the Umbral Calculus, *Integral Transforms and Special Functions*, **14**(2) , (2003), 181-186.
7. W. H. Ford, T. W. Ting, Stability and Convergence of Difference Approximations to Pseudo Parabolic Partial Differential Equations, *Mathematics of Computation*, **27**(124), (1973), 739-743.
8. H. Halilov, I. Ciftci, Fourier Method for a Qusilinear Pseudo-Parabolic Equation with Periodic Boundary Condition, *Hacettepe Journal of Mathematics and Statistics*, **37**(2), (2008), 69-79.
9. F. Kanca, I. Baglan, Determination of an Unknown Heat Source from Integral Overdetermination Condition, *Iranian Journal of Science and Technology, Transaction A: Science*, **42**(1), (2018), 1373-1382.
10. F. Kanca, I. Baglan, Inverse Problem for Euler-Bernoulli Equation with Periodic Boundary Condition, *Filomat*, **32**(16), (2018), 5691–5705.
11. F. Kanca, I. Baglan, Analysis for Two-Dimensional Inverse Quasilinear Parabolic Problem by Fourier Method, *Inverse Problems in Science and Engineering*, **29**(12), (2021), 1912–1945.
12. K. H. Kwek, C. C. Qu, Alternative Principle for Pseudo-Parabolic Equations, *Dynamic Systems and Applications*, **5**(2), (1996), 211–217.
13. W. Luther, Control Problems Governed by a Pseudo-Parabolic Partial Differantial Equation, *American Mathematical Society*, **250**, (1979), 235-246.
14. R. Pourgholia, M. Rostamiana and M.Emamjome, A Numerical Method for Solving a Nonlinear Inverse Parabolic Problem, *Inverse Problems in Science and Engineering*, **18**(8), (2010), 1151-1164.
15. S. Showalter, T.W. Ting, Pseudoparabolic Partial Differential Equations, *SIAM Journal on Mathematical Analysis*, **1**, (1970), 1–26.
16. T. W. Ting, Parabolic and Pseudo-Parabolic Partial Partial Differential Equations, *Journal of the Mathematical Society of Japan*, **21**, (1969), 440–453.