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Repdigits as Products of Consecutive Pell or Pell–Lucas Numbers

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Abstract. A positive integer is called a repdigit if it has only one distinct digit in its decimal expansion. In this paper, we find all repdigits that are products of consecutive Pell or Pell–Lucas numbers. This paper continues previous work which dealt with finding occurrences of repdigits in the Pell and Pell–Lucas sequences.

Keywords: Pell and Pell–Lucas number, Repdigit, Linear form in logarithms, Reduction method.

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1. Introduction

Given coprime integers s and t with $d := s^2 - 4t \neq 0$, the *Lucas sequence of the first kind* $(U_n)_{n\geq 0} = (U_n(s,t))_{n\geq 0}$ is a recursive sequence defined by

$$U_n = sU_{n-1} - tU_{n-2} \quad \text{for all} \quad n \geq 2, \tag{1.1}$$

with $U_0 = 0$ and $U_1 = 1$ as initial conditions. The *Lucas sequence of the second kind* $(V_n)_{n\geq 0} = (V_n(s,t))_{n\geq 0}$ satisfies the same linear recurrence relation but with initial values $V_0 = 2$ and $V_1 = s$.

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If $(\alpha, \beta) = ((s+\sqrt{d})/2, (s-\sqrt{d})/2)$ is the pair of roots of the characteristic equation $X^2 - sX + t = 0$ of both $(U_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$, then the Binet–type formulas for their general terms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n \text{ for all } n \geq 0. \tag{1.2}$$

If $(s,t) = (1,-1)$, then $(U_n)_{n\geq 0}$ is the Fibonacci sequence $(F_n)_{n\geq 0}$ and $(V_n)_{n\geq 0}$ is the Lucas sequence $(L_n)_{n\geq 0}$. Fibonacci and Lucas sequences are included in *Sloane's On–Line Encyclopedia of Integer Sequences* (OEIS) [14] as A000045 and A000032, respectively.

A positive integer is called a *repdigit* if it has only one distinct digit in its decimal expansion. The sequence of numbers with repeated digits appears as A010785 in OEIS.

The problem of looking for repdigits in a linearly recurrent sequence has been studied by several authors. For example, in 2000, Luca [9] showed that 55 and 11 are the only repdigits in the Fibonacci and Lucas sequences, respectively. Marques [10] looked for repdigits in the Tribonacci sequence (or 3–Fibonacci) and proved that 44 is the largest such. More generally, Bravo and Luca [4] extended the results mentioned above by showing that the only repdigits with at least two digits in the family of k–Fibonacci sequences occur only for $k = 2, 3$, proving a conjecture raised by Marques [10]. The k–Fibonacci sequence is defined similarly to the Fibonacci, but starting with $0, 0, \ldots, 0, 1$ (a total of k terms), with each term thereafter the sum of the k preceding terms.

On the other hand, with $(s,t) = (2,-1)$ we obtain that $(U_n)_{n\geq 0} = (P_n)_{n\geq 0}$ is the Pell sequence (OEIS A000129) and $(V_n)_{n\geq 0} = (Q_n)_{n\geq 0}$ is the Pell–Lucas sequence (OEIS A002203). Regarding these sequences and repdigits, in 2015, Faye and Luca [7] proved that there are no Pell or Pell–Lucas numbers larger than 10 with only one distinct digit.

Recently, the problem of determining the presence of repdigits in some Lucas sequences has been extended to study the product of consecutive terms which are repdigits. For instance, Marques and Togbé [11] showed that no product of two or more consecutive Fibonacci numbers can be a repdigit with at least two digits, while Irmak and Togbé [8] proved that 77 is the only repdigit with at least two digits occurring as the product of two or more consecutive Lucas numbers. Moreover, Erduvan, Keskin and Şiar [6] proved that no product of two Pell or Pell–Lucas numbers can be a repdigit with at least two digits.

In this paper, we look for repdigits in the products of consecutive Pell and Pell–Lucas numbers. More precisely, we prove the following results.

Theorem 1.1. *The only solutions of the Diophantine equation*

$$P_n P_{n+1} \cdots P_{n+\ell-1} = c \cdot \frac{10^m - 1}{9}, \tag{1.3}$$

in positive integers n, ℓ, c, m *with* $1 \leq c \leq 9$ *are*

$$(n, \ell, c, m) \in \{(1,1,1,1), (1,2,2,1), (2,1,2,1), (3,1,5,1)\}.$$

Theorem 1.2. *The only solutions of the Diophantine equation*

$$Q_n Q_{n+1} \cdots Q_{n+\ell-1} = c \cdot \frac{10^m - 1}{9}, \tag{1.4}$$

in nonnegative integers n, ℓ, c, m *with* $\ell \geq 1$, $m \geq 1$ *and* $1 \leq c \leq 9$ *are*

$$(n, \ell, c, m) \in \{(0,1,2,1), (0,2,4,1), (1,1,2,1), (2,1,6,1)\}.$$

Our method to prove both theorems is similar to the one employed by Bravo, Gómez and Luca [2]. First, we use the 2–adic order of the Pell and Pell–Lucas numbers to bound the number of factors ℓ. Next, we use linear forms in logarithms to find an upper bound for $\max\{m, n\}$. Finally, we use a Baker–Davenport reduction method (Lemma 2.4) to reduce such bounds to a manageable set of values which may then be verified directly using a computer.

2. Auxiliary results

2.1. **The Pell and Pell–Lucas sequences.** For a prime number p and a nonzero integer r, the p–adic order $\upsilon_p(r)$ is the exponent of the highest power of the prime p which divides r. The p–adic order of Lucas sequences of the first kind was completely characterized by Sanna [13].

Theorem 2.1. *Let* $(U_n)_{n\geq 0}$ *be a Lucas sequence given by* (1.1)*, and let* p *be a prime number such that* $p \nmid t$*. For each positive integer* n*, we have*

$$\upsilon_p(U_n) = \begin{cases} \upsilon_p(n) + \upsilon_p(U_p) - 1, & \text{if } p \mid d,\ p \mid n; \\ 0, & \text{if } p \mid d,\ p \nmid n; \\ \upsilon_p(n) + \upsilon_p(U_{p\tau(p)}) - 1, & \text{if } p \nmid d,\ \tau(p) \mid n,\ p \mid n; \\ \upsilon_p(U_{\tau(p)}), & \text{if } p \nmid d,\ \tau(p) \mid n,\ p \nmid n; \\ 0, & \text{if } p \nmid d,\ \tau(p) \nmid n, \end{cases}$$

where $\tau(p)$ *is the least positive integer such that* $p \mid U_{\tau(p)}$.

With $p = 2$, $(s,t) = (2,-1)$ and $d = 8$, we extract the 2–adic order of a Pell number; namely $\upsilon_2(P_n) = \upsilon_2(n)$ for all $n \geq 1$. On the other hand, it is a simple matter to show via induction that $\upsilon_2(Q_n) = 1$ for all $n \geq 1$. Let us state these properties as a lemma.

Lemma 2.2. *For all* $n \geq 1$,

(*a*) $\upsilon_2(P_n) = \upsilon_2(n)$*, and*
(*b*) $\upsilon_2(Q_n) = 1$.

On the other hand, using (1.1) with $(s,t)=(2,-1)$ and induction on n, it is easy to prove that the inequalities

$$P_n \leq \alpha^{n-1} \text{ and } Q_n < 2\alpha^n \text{ hold for all } n \geq 1, \tag{2.1}$$

where $\alpha := 1+\sqrt{2}$. Additionally, in view of (1.2), we can write

$$P_n = \frac{\alpha^n}{2\sqrt{2}} + e_n \text{ and } Q_n = \alpha^n + k_n \text{ where } |e_n| < 1/5 \text{ and } |k_n| < 1/2, \tag{2.2}$$

for all $n \geq 1$.

2.2. **Linear forms in logarithms.** In order to prove our main results, we need to use a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers. We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree k with minimal primitive polynomial

$$a_0 \prod_{i=1}^{k} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{i=1}^{k} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right)$$

is called the *logarithmic height* of η. Regarding this height, it is well known that

(*a*) $h(\eta) = \log\max\{|p|, q\}$, if $\eta = p/q \in \mathbb{Q}$ with $\gcd(p,q)=1$ and $q>0$;
(*b*) $h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2$;
(*c*) $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$;
(*d*) $h(\eta^u) = |u| h(\eta)$, $u \in \mathbb{Z}$.

The above facts about the logarithmic height will be used in the next sections without special reference. Now, we are ready to state the following general lower bound for a nonzero linear form in t logarithms due to Matveev [12].

Theorem 2.3. *Let $\mathbb{L}$ be a number field of degree $d_{\mathbb{L}}$ over $\mathbb{Q}$, $\eta_1, \ldots, \eta_t$ be positive real numbers of $\mathbb{L}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1.$$

Let $A_1, \ldots A_t$ be positive real numbers such that

$$A_i \geq \max\{d_{\mathbb{L}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log B) A_1 \cdots A_t).$$

2.3. **Reduction lemma.** To lower the upper bounds resulting from linear forms in logarithms, we will use the following lemma which is a slight variation of a result due to Dujella and Pethő [5] and itself is a generalization of a result of Baker and Davenport [1]. We shall use the version given by Bravo, Gómez and Luca in [3].

Lemma 2.4. *Let M be a positive integer and p/q be a convergent of the continued fraction of the real number κ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution of the inequality*

$$0 < |r\kappa - s + \mu| < AB^{-w}$$

in positive integers r, s and w with

$$r \le M \quad \text{and} \quad w \ge \frac{\log(Aq/\epsilon)}{\log B}.$$

3. Proof of Theorem 1.1

3.1. **On the number ℓ of factors.** We claim that $\ell \le 7$. Indeed, if $n \equiv 0 \pmod 2$, then $n+i \equiv 0 \pmod 2$ for $i = 0, 2, 4, 6$. So, by Lemma 2.2 (a), we get $\upsilon_2(P_{n+i}) = \upsilon_2(n+i) \ge 1$ for $i = 0, 2, 4, 6$. Consequently $\upsilon_2(P_n P_{n+1} \cdots P_{n+6}) \ge 4$. Now, if $n \equiv 1 \pmod 2$, then $n + j \equiv 0 \pmod 2$ for $j = 1, 3, 5, 7$, and so $\upsilon_2(P_{n+j}) = \upsilon_2(n+j) \ge 1$ for $j = 1, 3, 5, 7$. Hence $\upsilon_2(P_n P_{n+1} \cdots P_{n+7}) \ge 4$. Finally, since $\upsilon_2(c(10^m - 1)/9) = \upsilon_2(c) \le 3$, it follows from (1.3) that $\ell \le 7$.

3.2. **An absolute upper bound for m and n.** First of all, suppose that $n \ge 6$. Combining (1.3) and the left–hand side of (2.1), we obtain

$$10^{m-1} < \alpha^{\ell n + (\ell-3)(\ell-2)/2} < \alpha^{\ell n + \ell(\ell-1)/2},$$

so

$$m < \ell n + \ell(\ell-1)/2. \tag{3.1}$$

Now, using (2.2), we get

$$\begin{aligned} P_n \cdots P_{n+\ell-1} &= \left(\frac{\alpha^n}{2\sqrt{2}} + e_n\right) \cdots \left(\frac{\alpha^{n+\ell-1}}{2\sqrt{2}} + e_{n+\ell-1}\right) \\ &= (2\sqrt{2})^{-\ell} \alpha^{\ell n + \ell(\ell-1)/2} + r(\alpha, \ell, n), \end{aligned}$$

where $r(\alpha, \ell, n)$ is the sum of the remaining 127 terms of the expansion of the previous line that contains the product of powers of $(2\sqrt{2})^{-1}$, α and the errors e_i, for $i = n, \ldots, n + \ell - 1$. By a simple inspection, it is not difficult to see that the maximum of the absolute values of the terms appearing in $r(\alpha, \ell, n)$ is $5^{-1}(2\sqrt{2})^{1-\ell}\alpha^{(\ell-1)n + \ell(\ell-1)/2}$.

Combining now the above equality with (1.3), we obtain

$$(2\sqrt{2})^{-\ell}\alpha^{\ell n + \ell(\ell-1)/2} - \frac{c}{9}10^m = -\frac{c}{9} - r(\alpha, \ell, n),$$

and dividing across by $(2\sqrt{2})^{-\ell}\alpha^{\ell n+\ell(\ell-1)/2}$, we get that

$$\left|\frac{c(2\sqrt{2})^{\ell}}{9}\alpha^{-(\ell n+\ell(\ell-1)/2)}10^{m}-1\right| \le \left(\frac{c}{9}+|r(\alpha,\ell,n)|\right)\cdot(2\sqrt{2})^{\ell}\alpha^{-(\ell n+\ell(\ell-1)/2)} < 77\alpha^{-n}. \tag{3.2}$$

In order to use the result of Matveev Theorem 2.3, we take $t := 3$ and

$(\eta_1, b_1) := (c(2\sqrt{2})^{\ell}/9, 1)$, $(\eta_2, b_2) := (\alpha, -(\ell n+\ell(\ell-1)/2))$, $(\eta_3, b_3) := (10, m)$.

The number field containing η_1, η_2, η_3 is $\mathbb{L} := \mathbb{Q}(\alpha)$ which has degree $d_{\mathbb{L}} := 2$. We claim that the left–hand side of (3.2) is not zero. Indeed, if this were zero, we would then get that

$$\alpha^{\ell n+\ell(\ell-1)/2} = \frac{c\cdot 10^{m}}{9}(2\sqrt{2})^{\ell},$$

and so $\alpha^{2\ell n+\ell(\ell-1)} = c^2\cdot 10^{2m}\cdot 8^{\ell}/81 \in \mathbb{Q}$, which is not possible. From the properties of the logarithmic height, we have $h(\eta_1) \le h(c/9)+h((2\sqrt{2})^{\ell}) \le \log 9+\ell\log(2\sqrt{2}) < 19/2$, $h(\eta_2) = (\log\alpha)/2$ and $h(\eta_3) = \log 10$, so we can take $A_1 := 19$, $A_2 := \log\alpha$ and $A_3 := 2\log 10$. According to (3.1), we can take $B := \ell n+\ell(\ell-1)/2$. Then, Matveev's theorem gives

$$\left|\frac{c(2\sqrt{2})^{\ell}}{9}\alpha^{-(\ell n+\ell(\ell-1)/2)}10^{m}-1\right| > \exp\left(-7.5\cdot 10^{13}(1+\log(\ell n+\ell(\ell-1)/2))\right).$$

By comparing this with (3.2) and recalling that $\ell \le 7$, we get

$$n\log\alpha-\log 77 < 7.5\cdot 10^{13}(1+\log(7n+21)),$$

giving $n < 9\cdot 10^{13}(1+\log(7n+21))$. Thus $n < 4\cdot 10^{15}$. We record what we have proved so far as a lemma.

Lemma 3.1. *If (n,ℓ,c,m) is a positive integer solution of (1.3) with $n \ge 6$, then $1 \le \ell \le 7$ and*

$$n < 4\cdot 10^{15}.$$

3.3. **Reducing n.** Let $\Gamma := m\log 10-(\ell n+\ell(\ell-1)/2)\log\alpha+\log(c(2\sqrt{2})^{\ell}/9)$. Therefore, (3.2) can be rewritten as $|e^{\Gamma}-1| < 77/\alpha^{n}$. Note that $|e^{\Gamma}-1| < 1/2$ for all $n \ge 6$ (because $77/\alpha^{n} < 1/2$ for all $n \ge 6$). If $\Gamma > 0$, then $0 < \Gamma \le e^{\Gamma}-1 < 77/\alpha^{n}$. If, on the contrary, $\Gamma < 0$, then $e^{|\Gamma|} < 2$, and so $0 < |\Gamma| \le e^{|\Gamma|}-1 = e^{|\Gamma|}|e^{\Gamma}-1| < 154/\alpha^{n}$. Hence, $0 < |\Gamma| < 154/\alpha^{n}$ holds for $n \ge 6$.

Replacing Γ in the above inequality by its formula and dividing both sides of the resulting inequality by $\log\alpha$, we obtain

$$0 < |m\kappa-(\ell n+\ell(\ell-1)/2)+\mu| < 180\alpha^{-n}, \tag{3.3}$$

where $\kappa := \log 10/\log\alpha$ and $\mu := \log(c(2\sqrt{2})^{\ell}/9)/\log\alpha$. Put $M := 2.81\cdot 10^{16}$, which is an upper bound on m by (3.1) and Lemma 3.1. Applying Lemma 2.4 to the inequality (3.3) for each $1\le\ell\le 7$ and $1\le c\le 9$, we obtain that

$$n < \frac{\log(Aq/\epsilon)}{\log B},$$

where $q > 6M$ is a denominator of a convergent of the continued fraction of κ such that $\epsilon := \|\mu q\| - M\,\|\kappa q\| > 0$. A simple routine in *Mathematica* revealed that if $1\le\ell\le 7$ and $1\le c\le 9$, then the maximum value of $\log(Aq/\epsilon)/\log B$ is ≤ 60. Thus $n\le 60$. Consequently, we just need to check equation (1.3) in the range $1\le n\le 60$ and $1\le\ell\le 7$. For this, we used *Mathematica* and found that the only solutions of (1.3) are given in the statement of Theorem 1.1. This ends the proof of Theorem 1.1.

4. The proof of Theorem 1.2

Assume that $n\ge 3$. First of all, by Lemma 2.2 (b), we get that the 2–adic order of the left–hand side of (1.4) is ℓ, whereas its right–hand side has 2–adic order $\upsilon_2(c)\le 3$, so $\ell\le 3$. On the other hand, by using arguments similar to those used to obtain (3.1) and (3.2), one gets that

$$m \le \ell(n+1) + \ell(\ell-1)/2, \tag{4.1}$$

and

$$\left|\frac{c}{9}\alpha^{-(\ell n+\ell(\ell-1)/2)}10^{m} - 1\right| < 5\alpha^{-n}. \tag{4.2}$$

Here, Theorem 2.3 tells us that $\exp(-1.75\cdot 10^{13}(1+\log(\ell(n+1)+\ell(\ell-1)/2)))$ is a lower bound for the left–hand side of (4.2). Comparing this lower bound with (4.2), and then taking logarithms in both sides of the resulting inequality, we get that $n < 2\cdot 10^{13}(1+\log(3n+6))$.

Lemma 4.1. *If (n,ℓ,c,m) is a positive integer solution of (1.4) with $n\ge 3$, then $1\le\ell\le 3$ and*

$$n < 8\cdot 10^{14}.$$

In order to lower the upper bound on n from Lemma 4.1, we argue as in (3.3) to obtain

$$|m\kappa - (\ell n + \ell(\ell-1)/2) + \mu| < AB^{-n}, \tag{4.3}$$

where now $\kappa := \log 10/\log\alpha$, $\mu := \log(c/9)/\log\alpha$, $A := 12$ and $B := \alpha$. Here, we took $M := 2.41\cdot 10^{15}$, which is an upper bound on m by (4.1) and Lemma 4.1, and we applied Lemma 2.4 to the inequality (4.3) for each $1\le\ell\le 3$ and $1\le c\le 8$. In this case with the help of *Mathematica*, we found that the maximum value of $\log(Aq/\epsilon)/\log B$ is ≤ 50. Thus $n\le 50$. We cannot study the case $c = 9$ as before because when applying Lemma 2.4 to the expression (4.3), the corresponding parameter μ appearing there is 0 which yields that the corresponding value of ϵ from Lemma 2.4 is negative. Therefore, the reduction method is not useful for reducing the bound on n in this instance. However, when $c = 9$ we deduce that the 2–adic order of the right–hand side of (1.4) is 1, and hence $\ell = 1$, which is a case already studied by Faye and Luca [7].

Finally, a computer search with *Mathematica* for the range $0 \leq n \leq 50$ and $1 \leq \ell \leq 3$ revealed that the only solutions of (1.4) are those given in Theorem 1.2. This completes the proof of Theorem 1.2.

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