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On Local Antimagic Chromatic Number of Graphs with Cut-vertices

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Abstract. An edge labeling of a connected graph $G = (V, E)$ is said to be local antimagic if it is a bijection $f : E \to \{1, \dots, |E|\}$ such that for any pair of adjacent vertices x and y, $f^+(x) \neq f^+(y)$, where the induced vertex label $f^+(x) = \sum f(e)$, with e ranging over all the edges incident to x. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of distinct induced vertex labels over all local antimagic labelings of G. In this paper, the sharp lower bound of the local antimagic chromatic number of a graph with cut-vertices given by pendants is obtained. The exact value of the local antimagic chromatic number of many families of graphs with cut-vertices (possibly given by pendant edges) are also determined. Consequently, we partially answered Problem 3.1 in [Local antimagic vertex coloring of a graph, *Graphs and Combin.*, **33**, (2017), 275–285].

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1. Introduction

A connected graph $G = (V, E)$ is said to be *local antimagic* if it admits a *local antimagic edge labeling*, i.e., a bijection $f : E \to \{1, \dots, |E|\}$ such that the induced vertex labeling $f^+ : V \to \mathbb{Z}$ given by $f^+(u) = \sum f(e)$ (with e ranging over all the edges incident to u) has the property that any two adjacent vertices have distinct induced vertex labels. Thus, f^+ is a coloring of G. Clearly, the order of G must be at least 3. The vertex label $f^+(u)$ is called the *induced color* of u under f (the *color* of u, for short, if no ambiguous occurs). The number of distinct induced colors under f is denoted by $c(f)$, and is called the *color number* of f. The *local antimagic chromatic number* of G, denoted by $\chi_{la}(G)$, is $\min\{c(f) : f \text{ is a local antimagic labeling of } G\}$. Clearly, $2 \le \chi_{la}(G) \le |V(G)|$. The sharp lower bound of the local antimagic chromatic number of a graph with cut-vertices given by pendants is obtained. In [2, Problem 3.3], the authors asked:

Does there exist a graph G of order n with $\chi_{la}(G) = n - k$ for every $k = 0, 1, 2, \dots, n-2$?

In [3, Theorems 3.4 and 3.5], we proved the following that answered the above problem affirmatively.

Theorem 1.1. [3] *For each possible n, k, there exists a graph G of order n such that $\chi_{la}(G) = n - k$ if and only if $n \ge k + 3 \ge 3$. Moreover, there is a graph G of order n with $\chi_{la}(G) = 2$ if and only if $n \ne 2, 3, 4, 5, 7$.*

We shall in Section 2 completely determine the local antimagic chromatic number of the one-point union of cycles. Let G be a graph of order $n \ge 3$. We also determined the exact value of the local antimagic chromatic number of many families of graphs with pendants that has $\chi_{la}(G) < n$. In Section 3, we obtained several families of graphs G with $\chi_{la}(G) = n$. This partially answered [2, Problem 3.1]. For convenience, we shall use $a^{[n]}$ to denote a sequence of length n in which all items are a, where $n \ge 2$. For integers $1 \le a < b$, we let $[a, b]$ denote the set of integers from a to b.

2. $\chi_{la}(G) < |V(G)|$

In [2], the authors proved that for every tree T with k pendant edges (i.e., with k pendants), $\chi_{la}(T) \ge k+1$. We generalize this result to arbitrary graphs of order at least 3.

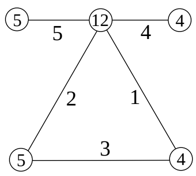
Lemma 2.1. *Let G be a graph of size q containing k pendants. Let f be a local antimagic labeling of G such that $f(e) = q$. If e is not a pendant edge, then $c(f) \geq k + 2$.*

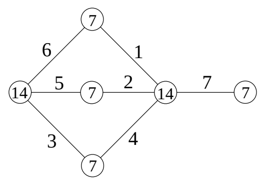
Proof. Let $e = uv$ and $x_1, \dots, x_k$ be pendants. Thus, $f^+(u) > q$ and $f^+(v) > q$ and they are distinct. On the other hand, $f^+(x_i) < q$ and are distinct for all i. Hence $c(f) \geq k + 2$. □

Theorem 2.2. *Let G be a graph having k pendants. If G is not K_2, then $\chi_{la}(G) \geq k + 1$ and the bound is sharp.*

Proof. Suppose G has size q. Let f be any local antimagic labeling of G. Consider the edge uv with $f(uv) = q$. We may assume u is not a pendant. Clearly, $f^+(u) > q \geq f^+(z)$ for every pendant z. Since all pendants have distinct induced colors, we have $\chi_{la}(G) \geq k + 1$.

For $k \geq 2$, since $\chi_{la}(S_k) = k + 1$, where S_k is a star with maximum degree k, the lower bound is sharp. The left labeling below is another example also showing that the lower bound is sharp. The right labeling shows that the lower bound is sharp for $k = 1$.





□

The contrapositive of the following lemma [3, Lemma 1] or [4, Lemma 2.1] gives a sufficient condition for a bipartite graph G to have $\chi_{la}(G) \geq 3$.

Lemma 2.3. [3, 4] *Let G be a graph of size q. Suppose there is a local antimagic labeling of G inducing a 2-coloring of G with colors x and y, where $x < y$. Let X and Y be the sets of vertices colored x and y, respectively, then G is a bipartite graph with bipartition (X, Y) and $|X| > |Y|$. Moreover, $x|X| = y|Y| = \frac{q(q+1)}{2}$.*

For $r \geq 2$ and $a_1 \geq a_2 \geq \cdots \geq a_r \geq 3$, denote by $C(a_1, a_2, \dots, a_r)$ the one-point union of r cycles of order $a_1, a_2, \dots, a_r$ respectively. Note that $C(a_1, a_2, \dots, a_r)$ has $m = a_1 + \cdots + a_r \geq 6$ edges and $m - r + 1$ vertices. We shall denote the vertex of maximum degree by u, called the *central vertex*, and the $2r$ edges incident to u are called the *central edges*. Denote the consecutive edges of subgraph C_{a_i} by $e_{s_i+1}, e_{s_i+2}, \dots, e_{s_i+a_i}$ such that $s_1 = 0$, $s_i = a_1 + a_2 + \cdots + a_{i-1}$ for $i \geq 2$. Moreover, for $i \geq 1$, e_{s_i+1} and $e_{s_i+a_i}$ are the central edges of C_{a_i}.

Theorem 2.4. *Suppose $G = C(a_1, a_2, \dots, a_r)$, then $\chi_{la}(G) = 2$ if and only if $G = C((4r-2)^{[r-1]}, 2r-2)$, $r \geq 3$ or $G = C((2r)^{[(r-1)/2]}, (2r-2)^{[(r+1)/2]})$, r is odd. Otherwise, $\chi_{la}(G) = 3$.*

Proof. Let $G = C(a_1, a_2, \ldots, a_r)$. Define an edge labeling $f : E(G) \to [1, m]$ by

1. $f(e_i) = i/2$ for even i,
2. $f(e_i) = m - (i-1)/2$ for odd i.

It is easy to verify that $f^+(u) > m+1$, and each vertex of degree 2 has color $m+1$ and m alternately beginning from vertices adjacent to u. Therefore, f is a local antimagic labeling that induces a 3-coloring. Thus, $\chi_{la}(G) \le 3$. If G contains an odd cycle, we have $\chi_{la}(G) \ge \chi(G) = 3$ so that $\chi_{la}(G) = 3$.

Suppose $\chi_{la}(G) = 2$. This implies that $\chi(G) = 2$ and hence $a_i \ge 4$ is even for each i. Let g be any local antimagic coloring of G that induces a 2-coloring of G with colors x and y. Without loss of generality, we may assume that $g^+(u) = y$. Let X and Y be the sets of vertices with colors x and y, respectively. It is easy to get that $|Y| = m/2 - r + 1$ and $|X| = m/2$. By Lemma 2.3, we have $x|X| = y|Y| = m(m+1)/2$. Hence, $x = m+1 \ge 4r+1$ is odd, $y = m(m+1)/(m-2r+2)$ and $y \ge 1 + 2 + \cdots + 2r = 2r^2 + r$. Suppose ℓ is labeled at an edge vw which is not a central edge. Without loss of generality, we may assume that $g^+(v) = x$ and $g^+(w) = y$. Then the label assigned to another edge incident with w must be $y - \ell$. Then $1 \le y - \ell \le m$, i.e., $\ell \ge y - m = y - x + 1$. In other word, labels in $[1, y-x]$ are labeled at central edges. So $y - x \le 2r$.

Solving for m, we get $m = (y - 1 \pm \sqrt{y^2 + 6y - 8yr + 1})/2$. Hence, $y^2 + 6y - 8yr + 1 = t^2 \ge 0$, where t is a nonnegative integer. This gives $(y+3-4r)^2 + 1 - (3-4r)^2 = t^2$ or $(y+3-4r-t)(y+3-4r+t) = 8(2r-1)(r-1)$. By letting $a = y+3-4r-t$ and $b = y+3-4r+t$ we have $2y+6-8r = a+b$ with $ab = 8(r-1)(2r-1) = 8(2r^2-3r+1)$. Clearly $b \ge a > 0$. Since a, b must be of same parity, we have both a, b are even.

Recall that $y - 2r^2 - r \ge 0$. Now

$$\begin{aligned} y - 2r^2 - r &= 4r - 3 + \frac{a+b}{2} - 2r^2 - r \\ &= \frac{a+b}{2} - 2r^2 + 3r - 3 = \frac{a+b}{2} - \frac{ab}{8} - 2 \\ &= \frac{4a + 4b - ab - 16}{8} = -\frac{(a-4)(b-4)}{8}. \end{aligned}$$

This implies that $a \le 4$.

Before considering the cases when $a \le 4$, we need the following claim which is easy to obtain.

Claim: *Let ϕ be a labeling of a $2s$-cycle $v_1v_2\cdots v_{2s}v_1$ with $\phi(v_{2i-1}v_{2i}) = \alpha_i$ and $\phi(v_{2i}v_{2i+1}) = \beta_i$ for $1 \le i \le s$, where $v_{2s+1} = v_1$. Suppose $\phi^+(v_{2j}) = x$ for $1 \le j \le s$ and $\phi^+(v_{2k+1}) = y$ for $1 \le k \le s-1$, where $y > x$. Then $\alpha_1 + \beta_1 = x$, $\{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ is an increasing sequence with common difference*

$y-x$ and $\{\beta_1, \beta_2, \ldots, \beta_s\}$ is an decreasing sequence with common difference $y-x$.

Case (1). Suppose $a = 2$. In this case, $b = 4(r-1)(2r-1)$ and $2y+6-8r = 8r^2-12r+6$. Hence, $y = 4r^2-2r$. This gives (i) $m = 4r^2-4r$ and $x = 4r^2-4r+1$ or (ii) $m = 2r-1$ and $x = 2r < 4r+1$, a contradiction. In (i), $y-x = 2r-1$. Since $[1, 2r-1]$ must be assigned to central edges, the central edges must be labeled by 1 to $2r-1$ and $2r^2-r$, respectively. There are $r-1$ cycles, say $C_{a_1}, \ldots, C_{a_{r-1}}$, whose central edges are labeled by numbers in $[1, 2r-1]$.

It is easy to verify that no such graph exists for $r = 2$. So we assume that $r \geq 3$.

Suppose C_{2s} is one of these $r-1$ cycles. Note that $s \geq 2$. Keep the notation defined in the claim. By symmetry, we may assume that $\alpha_1 < \beta_s$. So $\alpha_1 \in [1, 2r-2]$. Now we have $\beta_s = (x-\alpha_1)-(s-1)(y-x) \leq 2r-1$ and $\beta_{s-1} = (x-\alpha_1)-(s-2)(y-x) \geq 2r$. Thus, $(2r-1)^2-\alpha_1 \leq s(y-x) \leq 2r-2+(2r-1)^2-\alpha_1$. Hence, $(2r-1)^2-(2r-2) \leq s(2r-1) \leq (2r-1)^2+(2r-1)-2$. This implies that $2r-2 < s < 2r$. Thus $s = 2r-1$. Moveover, $\beta_s = \beta_{2r-1} = (2r-1)-\alpha_1 \leq 2r-2$. So, $a_j = 4r-2$ for $1 \leq j \leq r-1$.

We are now left with one unlabeled cycle, also denoted by C_{2s}, with central edge labels must be $2r-1$ and $2r^2-r$. Again, we may assume $2r-1 = \alpha_1 < \beta_s = 2r^2-r$. By the claim, $2r^2-r = (x-\alpha_1)-(s-1)(y-x) = (2r-1)^2-(2r-1)-(s-1)(2r-1)$. Thus $s = r-1$ and hence $a_r = 2r-2$. Therefore, $G = C((4r-2)^{[r-1]}, 2r-2)$.

On the other hand, for i-th $(4r-2)$-cycle, we choose $\alpha_1 = i$, $1 \leq i \leq r-1$; for the $(2r-2)$-cycle, we choose $\alpha_1 = 2r-1$. Apply the labeling as shown in the claim. One can verify that the edge labels are all distinct in $[1, 4r^2-4r]$. Consequently, $C((4r-2)^{[r-1]}, 2r-2)$ admits a local antimagic labeling that induces a 2-coloring. Thus, $\chi_{la}(C((4r-2)^{[r-1]}, 2r-2)) = 2$.

Case (2). Suppose $a = 4$. In this case, $b = 2(r-1)(2r-1)$ and $2y+6-8r = 4r^2-6r+6$. Hence, $y = 2r^2+r$. This gives (i) $m = 2r^2-r-1$ and $x = 2r^2-r$ with r is odd or (ii) $m = 2r$ and $x = 2r+1 < 4r+1$, a contradiction. In (i), $y-x = 2r$. Thus all the central edges must be assigned with integers in $[1, 2r]$. Suppose C_{2s} is one of the cycles whose central edges are labeled by α_1 and β_s. Also, by symmetry we may assume $\alpha_1 < \beta_s$. So $\alpha_1 \in [1, 2r-1]$. By a similar computation as in Case (1), we have $r - \frac{3}{2} + \frac{1}{r} \leq s \leq r + \frac{1}{2} - \frac{1}{r}$. So $s = r-1$ or r. Suppose there are k cycles of $2r$ edges in G, then there are $r-k$ cycles of $2r-2$ edges. Now, the size of G is $m = 2rk+(2r-2)(r-k) = 2r^2-2r+2k$. Thus we have $k = \frac{r-1}{2}$. Hence $G = C((2r)^{[(r-1)/2]}, (2r-2)^{[(r+1)/2]})$.

Moreover, when $s = r$ and since $\alpha_1 < \beta_s$, we have $\alpha_1 \leq r/2$. Since r is odd, $\alpha_1 \leq \frac{r-1}{2}$. Thus labels in $[1, \frac{r-1}{2}]$ are labeled at each of $2r$-cycle, respectively.

On the other hand, for i-th $2r$-cycle, we choose $\alpha_1 = i$, $1 \leq i \leq \frac{r-1}{2}$; for the j-th $(2r-2)$-cycle we choose $\alpha_1 = j + \frac{r-1}{2}$, $1 \leq j \leq \frac{r+1}{2}$. Apply the labeling as shown in the claim. One may verify that the edge labels are all distinct in $[1, 2r^2 - r - 1]$. Consequently, $C((2r)^{[(r-1)/2]}, (2r-2)^{[(r+1)/2]})$ admits a local antimagic labeling that induces a 2-coloring. Thus,

$$\chi_{la}(C((2r)^{[(r-1)/2]}, (2r-2)^{[(r+1)/2]})) = 2.$$

Consequently, $\chi_{la}(G) = 2$ if and only if $G = C((4r-2)^{[r-1]}, 2r-2), r \geq 3$ or $G = C((2r)^{[(r-1)/2]}, (2r-2)^{[(r+1)/2]}), r$ is odd. Otherwise, $\chi_{la}(G) = 3$. □

EXAMPLE 2.5. For $C(10, 10, 4)$, beginning and ending with central edges, the two 10-cycles has consecutive labels 1, 24, 6, 19, 11, 14, 16, 9, 21, 4 and 2, 23, 7, 18, 12, 13, 17, 8, 22, 3 respectively while the 4-cycle has consecutive labels $5, 20, 10, 15$ with $y = 30$ and $x = 25$. For $C(14, 14, 14, 6)$, the three 14-cycles has consecutive edge labels 1, 48, 8, 41, 15, 34, 22, 27, 29, 20, 36, 13, 43, 6; 2, 47, 9, 40, 16, 33, 23, 26, 30, 19, 37, 12, 44, 5 and 3, 46, 10, 39, 17, 32, 24, 25, 31, 18, 38, 11, 45, 4 respectively, while the 6-cycle has consecutive edge labels $28, 21, 35, 14, 42, 7$ with $y = 56$, and $x = 49$. Similarly, for $C(6, 4, 4)$, the 6-cycle has consecutive edge labels $1, 14, 7, 8, 13, 2$ while the two 4-cycles has consecutive edge labels $3, 12, 9, 6$ and $4, 11, 10, 5$ respectively with $y = 21$ and $x = 15$. For $C(10, 10, 8, 8, 8)$, the two 10-cycles has consecutive labels 1, 44, 11, 34, 21, 24, 31, 14, 41, 4 and 2, 43, 12, 33, 22, 23, 32, 13, 42, 3 respectively, while the three 8-cycles has consecutive labels $5, 40, 15, 30, 25, 20, 35, 10$; 6, 39, 16, 29, 26, 19, 36, 9 and 7, 38, 17, 28, 27, 18, 37, 8 respectively, with $y = 55$ and $x = 45$.

For $k, r \geq 1$ and $a_1 \geq a_2 \geq \cdots \geq a_r \geq 3$, let $H(a_1, a_2, \ldots, a_r; k)$ be the *hibiscus graph* obtained by identifying the central of $C(a_1, a_2, \ldots, a_r)$ with an end-vertex of k copies of P_2. Clearly, $H(a_1, a_2, \ldots, a_r; k)$ has $m + k = a_1 + \cdots + a_r + k \geq 4$ edges and $m + k - r + 1$ vertices. For non-pendant vertices and edges, we shall adopt the notation of $C(a_1, a_2, \ldots, a_r)$ accordingly.

Theorem 2.6. *For $k \geq 1$,*

$$\chi_{la}(H(a_1, a_2, \ldots, a_r; k)) = \begin{cases} 3 & \text{if } k = 1, \\ k+1 & \text{if } k \geq 2. \end{cases}$$

Proof. Let v_j $(1 \leq j \leq k)$ be the pendant vertices of $G = H(a_1, a_2, \ldots, a_r; k)$. Define an edge labeling $f : E(G) \to [1, m+k]$ by

1. $f(e_i) = (i+1)/2$ for odd i,
2. $f(e_i) = m - i/2 + 1$ for even i,
3. $f(uv_j) = m + j$ for $1 \leq j \leq k$.

It is easy to verify that $f^+(u) > m + k + 3$, $f^+(v_j) = m + j$ for $1 \leq j \leq k$, and each degree 2 vertex has color $m+1$ and $m+2$ alternately beginning from vertices adjacent to u.

When $k \geq 2$, we have that f is a local antimagic labeling that induces a $(k+1)$-coloring so that $\chi_{la}(G) \leq k+1$. By Theorem 2.2, we know $\chi_{la}(G) \geq k+1$. Therefore, $\chi_{la}(G) = k+1$.

Suppose $k = 1$. Clearly, f is a local antimagic labeling that induces a 3-coloring. So $\chi_{la}(G) \leq 3$. If G contains an odd cycle, then $\chi_{la}(G) \geq \chi(G) = 3$. Hence, $\chi_{la}(G) = 3$. Suppose $\chi_{la}(G) = 2$. Then G is bipartite and hence a_i is even for each $1 \leq i \leq r$. Let g be a local antimagic coloring of G that induces a 2-coloring with colors x and y such that $g^+(u) = y$. By Lemma 2.1 $g(uv_1) = m+1$. Since $g^+(u) = y$, $g^+(v_1) = x$. Hence $x = m+1$.

Let X and Y be the sets of vertices with colors x and y, respectively. It is easy to get that $|Y| = (m-2r+2)/2$ and $|X| = (m+2)/2$. By Lemma 2.3, we have $y(m-2r+2)/2 = (m+1)(m+2)/2$. Hence, $y = (m+1)(m+2)/(m-2r+2)$.

Solving for m, we get $m = (y - 3 \pm \sqrt{y^2 + 2y - 8yr + 1})/2$. Hence, $y^2 + 2y - 8yr + 1 = t^2 \geq 0$. This gives $(y+1-4r)^2 - (1-4r)^2 + 1 = t^2$ or $(y+1-4r-t)(y+1-4r+t) = 8r(2r-1)$, where $t \geq 0$. By letting $a = y+1-4r-t$ and $b = y+1-4r+t$, we have $2y+2-8r = a+b$ with $ab = 8r(2r-1)$. Since a, b must be of same parity, we have both a, b are even.

Now, $y \geq m+1+\sum\limits_{i=1}^{2r} i \geq 4r+1+r(2r+1) = 2r^2+5r+1$. By a similar computation in the proof of Theorem 2.4, $0 < 2r \leq y - 2r^2 - 3r - 1 = -\frac{(b-4)(a-4)}{8}$. This implies that $a = 2$ and $b \geq 6$. In this case, $b = 4r(2r-1)$ and hence $y = 4r^2 + 2r$. Thus $t = 4r^2 - 2r - 1$ and hence $m = 4r^2 - 2$ or $m = 2r - 1$. Since $m \geq 4r$, $m = 4r^2 - 2$. Since $b \geq 6$, $r \geq 2$. Now $y = 4r^2 + 2r \geq (4r^2 - 1) + \sum\limits_{i=1}^{2r} i = 6r^2 + r - 1$ yields a contradiction. Thus $\chi_{la}(H(a_1, a_2, \ldots, a_r; 1)) = 3$. □

Let $T(m,n)$ be the vertex-gluing of the end vertex of a path P_m and a vertex of a cycle C_n. In some article, $T(m,n)$ is called a *tadpole graph.*

Theorem 2.7. *For $n \geq 3$, $m \geq 2$, $\chi_{la}(T(m,n)) = 3$.*

Proof. Note that $T(m,n)$ has order and size $m+n-1$. Let the edge set be $\{e_i = v_iv_{i+1} \mid i \in [1, m+n-2]\} \cup \{e_{m+n-1} = v_{m+n-1}v_m\}$ so that $v_i \in V(P_m)$ for $i \in [1,m]$ and $v_j \in V(C_n)$ for $j \in [m, m+n-1]$. Note that v_m is the vertex of degree 3. For $1 \leq i \leq m+n-1$, define an edge labeling $f : E(T(m,n)) \to [1, m+n-1]$ by

$$f(e_i) = \begin{cases} m+n-(i+1)/2 & \text{for odd } i, \\ i/2 & \text{for even } i. \end{cases}$$

We now have

$$f^+(v_m) = \begin{cases} \frac{3(m+n)}{2} & \text{for even } m, n, \\ \frac{3(m+n)-2}{2} & \text{for odd } m, n, \\ \frac{3(m+n)-3}{2} & \text{for odd } m \text{ and even } n, \\ \frac{3(m+n)-1}{2} & \text{for even } m \text{ and odd } n. \end{cases}$$

Moreover, for $i \neq m$, $f^+(v_i) = m + n - 1$ for odd i, $f^+(v_i) = m + n$ for even i. Thus, $\chi_{la}(T(m, n) \leq 3$.

Suppose there exists a local antimagic labeling f that induces a 2-coloring of $T(m, n)$ with colors x and y such that $x < y$. Then $T(m, n)$ is bipartite so that n is even. Let X and Y be the sets of vertices with colors x and y, respectively. Clearly $||X| - |Y|| \leq 1$. Combining with Lemma 2.3, we have $x|X| = (m + n)(m + n - 1)/2 = y|Y|$ and $|X| = |Y| + 1$. By Lemma 2.1, $f^+(v_1) = x = m + n - 1$. So $|X| = (m + n)/2$ and $|Y| = (m + n)/2 - 1$. Thus $y = (m + n)(m + n - 1)/(m + n - 2)$ which is not an integer, a contradiction. Thus, $\chi_{la}(T(m, n)) = 3$. □

For $a_1 \geq a_2 \geq \cdots \geq a_r \geq 3$, let $GB(a_1, a_2, \ldots, a_r)$ denote the generalized book graph which is the edge-gluing of cycles of order $a_i, 1 \leq i \leq r$, at a common edge. We shall denote this common edge by uv in the following three results.

Lemma 2.8. *For $r \geq 2$, $\chi_{la}(GB(a_1, a_2, \ldots, a_r)) \geq 3$.*

Proof. Let $G = GB(a_1, a_2, \ldots, a_r)$. Suppose G contains an odd cycle, then $\chi_{la}(G) \geq \chi(G) = 3$. Suppose G is bipartite, then G has the same size of parts. By the contrapositive of Lemma 2.3, we know $\chi_{la}(G) \neq 2$. Therefore, $\chi_{la}(G) \geq 3$. □

Theorem 2.9. *Suppose $r \geq 2$, we have $\chi_{la}(GB(3^{[r]})) = 3$.*

Proof. Let $G = GB(3^{[r]})$ such that $V(G) = \{u, v\} \cup \{x_i : 1 \leq i \leq r\}$ and $E(G) = \{uv\} \cup \{ux_i; 1 \leq i \leq r\} \cup \{vx_i : 1 \leq i \leq r\}$. Define a bijection $f : E(G) \to [1, 2r + 1]$ by

(i) $f(ux_i) = i$ for $1 \leq i \leq r$,
(ii) $f(vx_i) = 2r + 1 - i$ for $1 \leq i \leq r$,
(iii) $f(uv) = 2r + 1$.

It is easy to verify that $f^+(x_i) = 2r+1$ for $1 \leq i \leq r$, $f^+(u) = r(r+1)/2+2r+1$ and $f^+(v) = (r + 1)(3r + 2)/2$. Hence, f is a local antimagic labeling that induces a 3-coloring so that $\chi_{la}(G) \leq 3$. Since $\chi_{la}(G) \geq \chi(G) = 3$, we have $\chi_{la}(G) = 3$. □

If $GB(a_1, a_2, \ldots, a_r) \neq GB(3^{[r]})$, it is easy to get a local antimagic labeling that induces a 4-coloring.

Conjecture 2.1. *If* $a_1 \geq 4$*, then* $\chi_{la}(GB(a_1, a_2, \ldots, a_r)) = 4$.

Let $G(a_1, a_2, \ldots, a_r; m)$ be obtained by identifying the vertex u of $GB(a_1, a_2, \ldots, a_r)$ with a vertex of $m \geq 1$ copies of P_2.

Theorem 2.10. *Let* $G = G(3^{[r]}; m)$*, then*

$$\chi_{la}(G) = \begin{cases} 3 & \text{if } G = G(3^{[r]}; 1) \text{ or } G(3^{[2]}; 2), \\ 4 & \text{if } G = G(3^{[r]}; 2), r \geq 3, \\ m+1 & \text{if } m \geq \binom{r}{2} \geq 3, \\ m+2 & \text{if } 3 \leq m < \binom{r}{2}. \end{cases}$$

Proof. For non-pendant vertices, we adopt the notations of $GB(3^{[r]})$. The pendant vertices are denoted by $y_j, 1 \leq j \leq m$. By Theorem 2.2, we know $\chi_{la}(G) \geq m + 1$. Since G contains an odd cycle, we also have $\chi_{la}(G) \geq 3$. Suppose $m = 1$. Define a bijection $f : E(G) \to [1, 2r + 2]$ by

(i) $f(ux_i) = i$ for $1 \leq i \leq r$,
(ii) $f(vx_i) = 2r + 1 - i$ for $1 \leq i \leq r$,
(iii) $f(uy_1) = 2r + 1$,
(iv) $f(uv) = 2r + 2$.

Clearly, $f^+(u) = (r^2 + 9r + 6)/2 \neq f^+(v) = (3r^2 + 5r + 4)/2 \neq f^+(x_i) = 2r+1 = f^+(y_1)$. Thus, f is a local antimagic labeling that induces a 3-coloring. Thus, $\chi_{la}(G(3^{[r]}; 1)) = 3$.

Consider $m = 2$. Suppose f is a local antimagic labeling that induces a 3-coloring. Without loss of generality, we must have $(r + 1)(r + 2)/2 \leq f^+(v) = f^+(y_1) \leq 2r + 3$. Hence, $r = 2$. The labeling $f(uv) = 1, f(vx_1) = 2, f(vx_2) = 3, f(ux_1) = 5, f(ux_2) = 4, f(uy_1) = 6, f(uy_2) = 7$ gives $\chi_{la}(G(3^{[2]}; 2)) = 3$. For $r \geq 3$, we then have $\chi_{la}(G) \geq 4$. Define a bijection $f : E(G) \to [1, 2r + 3]$ by

(i) $f(ux_i) = 2r + 2 - i$ for $1 \leq i \leq r$,
(ii) $f(vx_i) = i$ for $1 \leq i \leq r$,
(iii) $f(uy_j) = 2r + 1 + j$ for $1 \leq j \leq 2$,
(iv) $f(uv) = r + 1$.

Clearly, f is a local antimagic labeling that induces a 4-coloring with $f^+(u) = (r + 2)(3r + 5)/2, f^+(v) = \binom{r+2}{2}$, $f^+(y_j) = 2r + 1 + j, j = 1, 2$ and $f^+(x_i) = 2r + 2, 1 \leq i \leq r$. Thus, $\chi_{la}(G(3^{[r]}; 2)) = 4$ for $r \geq 3$.

Consider $m \geq 3$. We have $\chi_{la}(G) \geq m + 1 \geq 4$. Suppose $m \geq \binom{r}{2}$. Define a bijection $f : E(G) \to [1, 2r + m + 1]$ by

(i) $f(ux_i) = 2r + 2 - i$ for $1 \leq i \leq r$,
(ii) $f(vx_i) = i$ for $1 \leq i \leq r$,
(iii) $f(uy_j) = 2r + 1 + j$ for $1 \leq j \leq m$,
(iv) $f(uv) = r + 1$.

Clearly, f is a local antimagic labeling that induces an $(m+1)$-coloring with $f^+(u) = (m+r+1)(3r+m+2)/2$, $f^+(v) = \binom{r+2}{2} = f^+(y_{\binom{r}{2}})$, $f^+(y_j) = 2r+1+j, j \in [2,m]\backslash\{\binom{r}{2}\}$ and $f^+(x_i) = 2r+2 = f^+(y_1), 1 \le i \le r$. Thus, $\chi_{la}(G(3^{[r]};m)) = m+1$ if $m \ge \binom{r}{2} \ge 3$.

Suppose $m < \binom{r}{2}$, then $r \ge 4$. If $\chi_{la}(G) = m+1$, we may assume that G admits a local antimagic labeling f with $f^+(y_1) = f^+(x_i) \ge r(2r+1)/2, f^+(y_2) = f^+(v) \ge \binom{r+2}{2}$. Observe that $f^+(y_j) \le 2r+1+m < 2r+1+\binom{r}{2} = \binom{r+2}{2} \le f^+(v)$ for $1 \le j \le m$, a contradiction. Thus, $\chi_{la}(G) \ge m+2$. Define a bijection $f : E(G) \to [1, 2r+m+1]$ by

(i) $f(ux_i) = i+1$ for $1 \le i \le r$,
(ii) $f(vx_i) = 2r+2-i$ for $1 \le i \le r$,
(iii) $f(uy_j) = 2r+1+j$ for $1 \le j \le m$,
(iv) $f(uv) = 1$.

It is easy to show that f is a local antimagic labeling that induces a 4-coloring with $f^+(u) = (r+1)(r+2)/2 + m(4r+m+3)/2$, $f^+(v) = 3r(r+1)/2+1$, $f^+(x_i) = f^+(y_2) = 2r+3, f^+(y_j) = 2r+1+j$ for $j = 1,3,4,\ldots,m$. Thus, $\chi_{la}(G(3^{[r]};m)) = m+2$ if $3 \le m < \binom{r}{2}$. □

Problem 2.1. *Study* $\chi_{la}(G(a_1,a_2,\ldots,a_r;m))$ *for* $a_1 \ge 4$.

Suppose G is of order m. Let $G \odot H$ be the graph obtained from G and m copies of H by joining the i-th vertex of G to each vertex of the i-th copy of H.

Let $G = C_m \odot O_n$ with $V(G) = \bigcup_{i=1}^{m}(\{v_{i,j} : 1 \le j \le n\} \cup \{u_i\})$ and $E(G) = \bigcup_{i=1}^{m}(\{u_iv_{i,j} : 1 \le j \le n\} \cup \{e_i\})$, where $e_i = u_iu_{i+1}$ for $1 \le i \le m$, and $u_{m+1} = u_1$ by convention. We shall keep these notation in the following discussion.

Lemma 2.11. *For* $m \ge 3$ *and* $n \ge 1$, $\chi_{la}(C_m \odot O_n) \ge mn+2$.

Proof. Let f be a local antimagic labeling of $G = C_m \odot O_n$. Let e be an edge of G such that $f(e) = m(n+1)$ which is the size of G. If e is not a pendant edge, then by Lemma 2.1, $c(f) \ge mn+2$. So we only need to deal with $e = u_iv_{i,j}$ for some $i \in [1,m]$ and $j \in [1,n]$. By renumbering we may assume that $e = u_1v_{1,1}$. Note that $f^+(v_{i,j}) \le m(n+1)$.

Suppose $f^+(u_i)$ and $f^+(u_{i+1})$ are greater than $m(n+1)$ for some i, $1 \le i \le m$. Since they are distinct, $c(f) \ge mn+2$. So we may assume that the induced colors of any two consecutive vertices of C_m do not both greater than $m(n+1)$. Let k be the number of vertices in C_m whose induced color is less than or equal to $m(n+1)$. Thus, $m-1 \ge k \ge \lceil m/2 \rceil$. All edges in the cycle C_m are incident to at least one of these k vertices. So there are exactly $m+kn$ distinct edges incident to these k vertices.

Now $(m+kn)(m+kn+1)/2 \le m(n+1)k$. Since $m \ge k+1$, $k(n+1)(m+kn+1) < [k(n+1)+1](m+kn+1) = (k+1+kn)(m+kn+1) \le 2m(n+1)k$.

Hence $m+kn+1<2m$ or $kn+1<m$. Since $k \geq \lceil m/2 \rceil$, $n=1$. For this case, we have $(m+k)(m+k+1) \leq 4mk$. This implies that $(m-k)^2+m+k \leq 0$ which is impossible. □

Theorem 2.12. *For $m \geq 3$ and $n \geq 1$, $\chi_{la}(C_m \odot O_n) = mn+2$ if both m and n are even; otherwise $mn+2 \leq \chi_{la}(C_m \odot O_n) \leq mn+3$.*

Proof. Suppose $G = C_m \odot O_n$. By Lemma 2.11, we know $\chi_{la}(G) \geq mn+2$.

Consider $m = 2h \geq 4$ and $n = 2k \geq 2$. Define $f : E(G) \to [1, 2h(2k+1)]$ by $f(e_i) = i$ for $1 \leq i \leq 2h$ and

$$\begin{aligned}
f(u_1v_{1,1}) &= 4h+1, & f(u_1v_{1,2}) &= 6h, \\
f(u_{2i-1}v_{2i-1,1}) &= 6h+3-2i, & f(u_{2i-1}v_{2i-1,2}) &= 6h+2-2i, \quad 2 \leq i \leq h; \\
f(u_{2i}v_{2i,1}) &= 4h+1-2i, & f(u_{2i}v_{2i,2}) &= 4h+2-2i, \quad 1 \leq i \leq h; \\
f(u_rv_{r,2j-1}) &= 2h(2j-1)+r, & & 1 \leq r \leq 2h, 2 \leq j \leq k; \\
f(u_rv_{r,2j}) &= 2h(2j+1)+1-r, & & 1 \leq r \leq 2h, 2 \leq j \leq k.
\end{aligned}$$

It is easy to check that f is a bijection. It is also easy to verify that all pendants have different colors from $2h+1$ to $2h(2k+1)$.

Now, for $2 \leq i \leq h$, we have

$$\begin{aligned}
f^+(u_{2i-1}) &= f(e_{2i-1}) + f(e_{2i-2}) + f(u_{2i-1}v_{2i-1,1}) + f(u_{2i-1}v_{2i-1,2}) \\
&\quad + \sum_{j=2}^{k} f(u_{2i-1}v_{2i-1,2j-1}) + \sum_{j=2}^{k} f(u_{2i-1}v_{2i-1,2j}) \\
&= (2i-1) + (2i-2) + (6h+3-2i) + (6h+2-2i) \\
&\quad + \sum_{j=2}^{k}[2h(2j-1)+(2i-1)] + \sum_{j=2}^{k}[2h(2j+1)+1-(2i-1)] \\
&= 12h+2+\sum_{j=2}^{k}[8hj+1] = 12h+2+(k-1)(4hk+8h+1).
\end{aligned}$$

We can also get that $f^+(u_1) = 12h+2+(k-1)(4hk+8h+1)$ and $f^+(u_{2i}) = 8h+2+(k-1)(4hk+8h+1)$ for $1 \leq i \leq h$. So we have $\chi_{la}(G) \leq mn+2$. Hence, $\chi_{la}(G) = mn+2$.

Consider odd m, n. Let $A = (a_{i,j})$ be a magic (m, n) rectangle involving the integers $[1, mn]$ (for the existence of magic rectangle, please see [5]). Let g be a local antimagic labeling of C_m with $c(g) = 3$. Now we define a labeling f for G by

$$f(e) = g(e) + mn \text{ for } e \text{ is an edge of } C_m;$$
$$f(u_iv_{i,j}) = a_{i,j}, \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

Clearly $c(f) = mn + 3$. So $\chi_{la}(G) \le mn + 3$.

Consider $m = 2h + 1 \ge 3$ and $n = 2k \ge 2$. Let α be the k-vector whose i-th coordinate is i, i.e., $\alpha = (1, 2, \dots, k)$. Let J be the k-vector whose coordinates are 1. Let B be a $(2h+1)\times 2k$ matrix whose i-th row is $((i-1)kJ+\alpha \ \ (4h+2-i)kJ+\alpha)$. Hence each row sum of B is $(4h+1)k^2 + k(k+1) = 4hk^2 + 2k^2 + k$. Note that B contains all integers in $[1, 4hk + 2k]$. Similar to the case of odd m, n, we will obtain a local antimagic labeling f of G with $c(f) = mn + 3$. So $\chi_{la}(G) \le mn + 3$.

Consider even $m = 2h \ge 4$ and odd $n = 2k - 1 \ge 1$. Let $X = \{jk : 1 \le j \le 2h\}$. We define a labeling $\phi : E(C_{2h}) \to X$ by $\phi(e_{2i-1}) = ik$ and $\phi(e_{2i}) = (h + i)k$, $1 \le i \le h$. Then $\phi^+(u_j) = (h + j)k$ for $2 \le j \le 2h$ and $\phi^+(u_1) = (2h + 1)k$.

Let C be a $(2h)\times(2k)$ matrix whose i-th row is $((4h-i)kJ+\alpha \ \ (i-1)kJ+\alpha)$. Hence each row sum of C is $N = (4h-1)k^2+k(k+1) = 4hk^2+k$. Let $C' = (c_{i,j})$ be a $(2h) \times (2k - 1)$ matrix obtained from C by deleting the last column of C. So the i-th row sum of C' is $N - ik$, $1 \le i \le 2h$. Now we shall label the pendant edges $u_iv_{i,j}$ by entries of a suitable row of C', $1 \le i \le 2h$ and $1 \le j \le 2k - 1$.

Let $\psi : E(G) \setminus E(C_{2h}) \to [1, 4hk] \setminus X$ defined by $\psi(u_{2i}v_{2i,j}) = c_{2i,j}$ for $1 \le i \le h$; $\psi(u_{2i+1}v_{2i+1,j}) = c_{2i-1,j}$, for $1 \le i \le h-1$; and $\psi(u_1v_{1,j}) = c_{2h-1,j}$, where $1 \le j \le 2k - 1$. Now $\psi^+(u_1) = N - (2h - 1)k$; $\psi^+(u_{2i}) = N - 2ik$ for $1 \le i \le h$ and $\psi^+(u_{2i+1}) = N - (2i - 1)k$ for $1 \le i \le h - 1$. Note that $\psi(u_3v_{3,k}) = 4hk$ which is the largest label.

Let f be the labeling of G obtained by combining ϕ and ψ. Hence
$f^+(u_1) = \phi^+(u_1) + \psi^+(u_1) = (2h + 1)k + [N - (2h - 1)k] = N + 2k$.
$f^+(u_{2i}) = \phi^+(u_{2i}) + \psi^+(u_{2i}) = (h + 2i)k + [N - 2ik] = N + hk$ for $1 \le i \le h$.
$f^+(u_{2i+1}) = \phi^+(u_{2i+1})+\psi^+(u_{2i+1}) = (h+2i+1)k+[N-(2i-1)k] = N+(h+2)k$ for $1 \le i \le h - 1$.

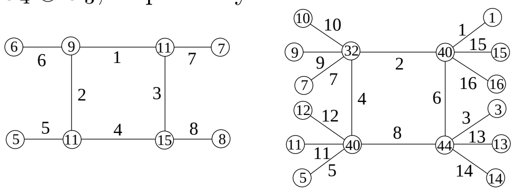
Here $c(f) = 2h(2k - 1) + 3$ if $h \ne 2$.

For $h = 2$, we redefine the labeling ϕ by $\phi(e_1) = k$, $\phi(e_2) = 3k$, $\phi(e_3) = 4k$ and $\phi(e_4) = 2k$. Then $\phi^+(u_1) = 3k$, $\phi^+(u_2) = 4k$, $\phi^+(u_3) = 7k$ and $\phi^+(u_4) = 6k$. Hence $f^+(u_1) = N$, $f^+(u_2) = N + 2k$, $f^+(u_3) = N + 6k$ and $f^+(u_4) = N + 2k$. Here $c(f) = 2h(2k - 1) + 3$ if $h = 2$.

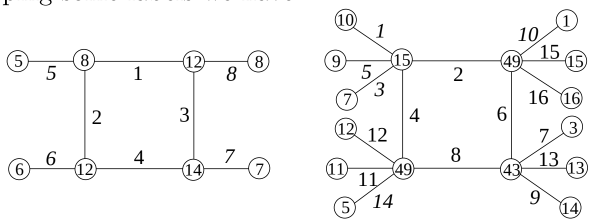
This completes the proof. □

Example 2.13. Consider $G = C_3 \odot O_1$. Denote $v_i = v_{i,1}$. Define $f(u_1u_2) = 2$, $f(u_2u_3) = 3$, $f(u_3u_1) = 4$, $f(u_1v_1) = 5$, $f(u_2v_2) = 1$ and $f(u_3v_3) = 6$. Then the colors of vertices are 1,5,6,11,13. So $\chi_{la}(G) \le 5$. By Theorem 2.12 we have $\chi_{la}(C_3 \odot O_1) = 5$.

Example 2.14. According to the proof of Theorem 2.12 we have the labelings for $C_4 \odot O_1$ and $C_4 \odot O_3$, respectively.



After swapping some labels we have



So $\chi_{la}(C_4 \odot O_1) \le 6$ and $\chi_{la}(C_4 \odot O_3) \le 14$. By Theorem 2.12 we have $\chi_{la}(C_4 \odot O_1) = 6$ and $\chi_{la}(C_4 \odot O_3) = 14$.

We are only aware, after obtaining the above theorem, that Arumugam et al. [1, Section 3] have also obtained partial solutions on $\chi_{la}(C_m \odot O_n)$. Particularly, their Lemmas 3.6, 3.12 and 3.13 imply that $\chi_{la}(C_m \odot O_n) = mn+2$ for even $m \ge 4$ and $n \ge 3$. Moreover, Lemmas 3.14, 3.15 and 3.16 partially solved the case when m is odd. This left $\chi_{la}(C_m \odot O_n)$ still unsolved for odd m and finitely many n.

3. $\chi_{la}(G) = |V(G)|$

For $m \ge 1$, $n_i \ge n_{i+1}$ $(1 \le i \le m-1)$, and $n_1 + n_2 + \cdots + n_m \ge 1$, let $K(m; n_1, n_2, \ldots, n_m)$ be obtained from K_m by joining n_i pendant vertices to the i-th vertex of K_m. Note that $K(2;1,0) \cong P_3$ with $\chi_{la}(P_3) = 3$ and $K(2;1,1) \cong P_4$ with $\chi_{la}(P_4) = 3$ (see [2, Theorem 2.7]). Moreover, $\chi_{la}(K(2;2,1)) = 4$ (see [3, Theorem 8]). Observe that $K(1;n-1) \cong K(2;n-2,0)$ is the star graph $K_{1,n-1}$ of order n with $\chi_{la}(K_{1,n-1}) = n$.

Theorem 3.1. *For $m \ge 2$, $\chi_{la}(K(2;n_1,0)) = n_1 + 2$. Otherwise, $\chi_{la}(K(m;n_1,n_2,\ldots,n_m)) \le n_1+n_2+\cdots+n_m+m$ and the equality holds if and only if $(n_m+m-1)(n_m+m)/2 > n_1+n_2+\cdots+n_m+\binom{m}{2}$.*

Proof. Note that $G = K(m;n_1,n_2,\ldots,n_m)$ has order $n = n_1+n_2+\cdots+n_m+m$ and size $q = n_1+n_2+\cdots+n_m+\binom{m}{2}$. By definition, it is easy to get $\chi_{la}(K(2;n_1,0)) = n_1+2$. We now assume $G \not\cong K(2;n_1,0)$. Let $V(G) = \{u_i : 1 \le i \le m\} \cup \{u_{i,k} : 1 \le i \le m, 1 \le k \le n_i\}$ and $E(G) = \{u_iu_j : 1 \le i < j \le$

$m\} \cup \{e_{i,k} = u_iu_{i,k} : 1 \le i \le m, 1 \le k \le n_i\}$. Suppose f is any local antimagic labeling of G. By definition, we must have all pendant vertex labels and all non-pendant vertex labels are mutually distinct respectively. Moreover, $f^+(u_i) \neq f^+(u_{i,k})$ and $f^+(u_{i,k}) \le q$ for all i, k. Since for $1 \le i \le m$, $\deg(u_i) \ge n_m+m-1$, we also have $f^+(u_i) \ge 1 + 2 + \cdots + (n_m + m - 1) = (n_m + m - 1)(n_m + m)/2$. Thus, if $(n_m + m - 1)(n_m + m)/2 > q$, we have $f^+(u_i) > f^+(u_{j,k})$ for all $1 \le i, j \le m, 1 \le k \le n_j$. Therefore, $\chi_{la} = n$. This prove the sufficiency.

To prove the necessity, suffice to show that if $(n_m + m - 1)(n_m + m)/2 \le q$, then $c(f) < n$ for some local antimagic labeling f. Consider the sequence of edges $S = u_mu_{m-1}, u_mu_{m-2}, u_mu_{m-3}, \ldots, u_mu_1, e_{m,1}, e_{m,2}, e_{m,3}, \ldots, e_{m,n_m}, u_{m-1}u_{m-2}, u_{m-1}u_{m-3}, \ldots, u_{m-1}u_1, e_{m-1,1}, e_{m-1,2}, e_{m-1,3}, \ldots, e_{m-1,n_{m-1}}, \ldots, u_2u_1, e_{2,1}, e_{2,2}, e_{2,3}, \ldots, e_{2,n_2}, e_{1,1}, e_{1,2}, \ldots, e_{1,n_1}$. Define $f : S \to [1, q]$ according to the order in S. Observe that

(i) f is bijective,
(ii) all the pendant vertex induced labels are distinct,
(iii) for $1 \le i \le m, 1 \le k \le n_i$, $f^+(u_i) > f^+(u_{i,k})$ if $n_i > 0$,
(iv) for $1 \le i \le m-1$, $f(u_iu_j) > f(u_{i+1}u_j)$ $(j \neq i, i+1)$ and $\sum_{k=1}^{n_i} f(e_{i,k}) > \sum_{k=1}^{n_{i+1}} f(e_{i+1,k})$ so that $f^+(u_i) > f^+(u_{i+1})$.

Thus, f is a local antimagic labeling. We now have $f^+(u_m) = 1 + 2 + \cdots + (n_m + m - 1) = (n_m + m - 1)(n_m + m)/2$ and $f^+(u_{i,k}) \le q$ for $1 \le i \le m, 1 \le k \le n_i$. Since $(n_m + m - 1)(n_m + m)/2 \le q$, then there exists an edge e with $f(e) = (n_m + m - 1)(n_m + m)/2$, where f is defined above. If $e = e_{i,k}$, then we have $f^+(u_{i,k}) = f^+(u_m)$ so that $c(f) < n$. Otherwise, since for $1 \le t \le m-1$, $f(u_tu_m) = m - t < m(m-1)/2 < (n_m + m - 1)(n_m + m)/2 = f(e)$, we must have $e = u_iu_j$ for $1 \le i < j \le m - 1$. We have the following two cases.

Case (a). $n_2 = 0$. In this case, $m \ge 3$. Note that K_m has size $\binom{m}{2} = 1 + 2 + \cdots + (m-1) = f^+(u_m)$. Thus, $e = u_2u_1$ and $e_{1,1}$ must be the next unlabeled pendant edge. We now swap the labels of u_2u_1 and $e_{1,1}$. It is easy to verify that a new local antimagic labeling g with $g^+(u_m) = g^+(u_{1,1}) = \binom{m}{2}$ is obtained. Therefore, $c(g) < n_1 + m = n$.

Case (b). $n_2 \neq 0$. In this case, according to our labeling sequence, $e_{j,1}$ must be the next unlabeled pendant edge. Let S' be obtained from S by putting $e_{j,1}$ right before e. Now, define $g : S' \to [1, q]$ according to the order in S'. One can verify that all the observations under f still hold under g. Moreover, $g^+(u_{j,1}) = g^+(u_m)$. Thus, g is a local antimagic labeling with $c(g) < n$. □

By Theorem 2.2, we know $\chi_{la}(K(2; a, b)) \ge a + b + 1$.

Corollary 3.2. *For $a \ge b \ge 2$,*

$$\chi_{la}(K(2; a, b)) = \begin{cases} a + b + 2 & \text{if } a < b(b+1)/2 \\ a + b + 1 & \text{otherwise.} \end{cases}$$

Suppose $k \geq 2$, $n_1, \ldots, n_k \geq 0$ and $n_1 + \cdots + n_k \geq 2$. Let $Ct(k; n_1, \ldots, n_k)$ be the caterpillar graph obtained from the path $P_k = v_1 v_2 \cdots v_k$ by joining n_i pendants to v_i. Consider the following two conditions:

C_1 : $\min\{(n_1+1)(n_1+2)/2, (n_2+2)(n_2+3)/2, (n_3+1)(n_3+2)/2\} > n_1 + n_2 + n_3 + 2$.

C_2 : No $n_1 + n_3 + 2$ of distinct integers in $[1, n_1 + n_2 + n_3 + 2]$ can have sum of the $n_1 + 1$ integers equal sum of the remaining $n_3 + 1$ integers.

Theorem 3.3. *Suppose $n_1, n_2, n_3 \geq 1$. If $Ct(3; n_1, n_2, n_3)$ satisfies conditions C_1 and C_2, then $\chi_{la}(Ct(3; n_1, n_2, n_3)) = n_1 + n_2 + n_3 + 3$.*

Proof. Let $G = Ct(3; n_1, n_2, n_3)$ be the caterpillar graph obtained from the path $P_3 = xyz$ by joining pendants $x_1, \ldots, x_{n_1}$ to x, pendants $y_1, \ldots, y_{n_2}$ to y and pendants $z_1, \ldots, z_{n_3}$ to z. Let f be a local antimagic labeling of G. Note that $f^+(x) \geq \frac{1}{2}(n_1+1)(n_1+2)$, $f^+(y) \geq \frac{1}{2}(n_2+2)(n_2+3)$ and $f^+(z) \geq \frac{1}{2}(n_3+1)(n_3+2)$. Moreover, $f^+(y) \neq f^+(x)$ and $f^+(y) \neq f^+(z)$. By C_2, we obtain that $f^+(x) \neq f^+(z)$. Combining the results above, by C_1 we have $\chi_{la}(G) \geq n_1 + n_2 + n_3 + 3$.

We now give a labeling $f : E(G) \to [1, n_1 + n_2 + n_3 + 2]$. By symmetry, we only need to consider three possibilities. Suppose $n_1 \leq n_2 \leq n_3$, we label in the sequence $xx_1, xx_2, \ldots, xx_{n_1}, xy, yy_1, yy_2, \ldots, yy_{n_2}, yz, zz_1, zz_2, \ldots, zz_{n_3}$. Clearly, $f^+(z) = (n_3+1)(2n_1+2n_2+4)/2 > f^+(y) = (n_2+2)(2n_1+n_2+3)/2 > f^+(x) = (n_1+1)(n_1+2)/2$ which in turn greater than all the pendant vertex labels. Thus, $c(f) = n_1 + n_2 + n_3 + 3$. Suppose $n_1 \leq n_3 < n_2$, we label in the sequence $xx_1, xx_2, \ldots, xx_{n_1}, xy, zz_1, zz_2, \ldots, zz_{n_3}, yz, yy_1, yy_2, \ldots, yy_{n_2}$. Similarly, we have $f^+(y) > f^+(z) > f^+(x)$ which in turn greater than all the pendant vertex labels. Thus, $c(f) = n_1 + n_2 + n_3 + 3$. Finally, suppose $n_2 < n_1 \leq n_3$, we label in the sequence

$$yy_1, yy_2, \ldots, yy_{n_2}, xy, xx_1, xx_2, \ldots, xx_{n_1}, zz_1, zz_2, \ldots, zz_{n_3}, yz.$$

Similarly, we have we have $f^+(z) > f^+(x) > f^+(y)$ which in turn greater than all the pendant vertex labels. Thus, $c(f) = n_1 + n_2 + n_3 + 3$. □

We are not able to find a $Ct(3; n_1, n_2, n_3)$ that satisfies Conditions C_1 and C_2. Thus, we have the following problem.

Problem 3.1. *Prove the existence of $Ct(3; n_1, n_2, n_3)$ in Theorem 3.3.*

Let us consider a special case $Ct(3; n_1, 0, n_3)$ with $n_3 \geq n_1$. In this case, C_1 does not hold. When $n_1 = 1$, $Ct(3; 1, 0, n_3)$ is a coconut graph. It is known that $\chi_{la}(Ct(3; 1, 0, n_3)) = n_3 + 2$. But we shall also include this result in the following corollary.

Corollary 3.4. *Suppose condition C_2 does not hold and $1 \leq n_1 \leq n_3$, then $\chi_{la}(Ct(3; n_1, 0, n_3)) = n_1 + n_3 + 1$.*

Proof. Let $Ct(3; n_1, 0, n_3)$ be as defined in Theorem 3.3. Since C_2 does not hold, $[1, n_1 + n_3 + 2]$ has a bipartition (S_1, S_3) such that $|S_1| = n_1 + 1$, $|S_3| = n_3 + 1$ and the sum of numbers in S_1 equals $\frac{1}{4}(n_1 + n_3 + 2)(n_1 + n_3 + 3)$ (of course, $n_1 + n_3 \equiv 1, 2 \pmod 4$).

Suppose $1 \in S_1$. Choose $a \in S_3$ arbitrarily. Define $f : E(Ct(3; n_1, 0, n_3)) \to [1, n_1+n_3+2]$ such that $f(xy) = 1, f(yz) = a$, $\{f(xx_i) \mid 1 \le i \le n_1\} = S_1 \setminus \{1\}$ and $\{f(zz_j) \mid 1 \le j \le n_3\} = S_3 \setminus \{a\}$. Now $f^+(x) = f^+(z) = \frac{1}{4}(n_1 + n_3 + 2)(n_1 + n_3 + 3)$, $f^+(y) = 1 + a$ which is equal to a label of a pendant vertex. So we obtain that $\chi_{la}(Ct(3; n_1, 0, n_3)) \le n_1 + n_3 + 1$. Similarly for $1 \in S_3$.

By Theorem 2.2 we have $\chi_{la}(Ct(3; n_1, 0, n_3)) = n_1 + n_3 + 1$. □

Corollary 3.5. *Suppose condition C_2 holds and $n_1 \le n_3 < (n_1+2)(n_1-1)/2$, then $\chi_{la}(Ct(3; n_1, 0, n_3)) = n_1 + n_3 + 2$.*

Proof. Under the assumption, $n_1 > 2$. Let $Ct(3; n_1, 0, n_3)$ be as defined in Theorem 3.3. Condition C_2 holds implies that $f^+(x) \ne f^+(z)$ for all possible local antimagic labeling f of $Ct(3; n_1, 0, n_3)$. Moreover, $n_3 < (n_1+2)(n_1-1)/2$ implies that $(n_3 + 1)(n_3 + 2)/2 \ge (n_1 + 1)(n_1 + 2)/2 > n_1 + n_3 + 2$ so that $f^+(x)$ and $f^+(z)$ are larger than all other pendant vertex colors for all possible local antimagic labeling of $Ct(3; n_1, 0, n_3)$. Thus, $c(f) \ge n_1 + n_3 + 2$.

Define $f : E(Ct(3; n_1, 0, n_3)) \to [1, n_1+n_3+2]$ such that $f(xy) = 1, f(yz) = 2$, $f(xx_i) = i + 2$ for $1 \le i \le n_1$, $f(zz_j) = n_1 + 2 + j$ for $1 \le j \le n_3$. Clearly, f is a local antimagic labeling with $n_1 + n_3 + 2$ distinct vertex colors. Thus, $\chi_{la}(Ct(3; n_1, 0, n_3)) = n_1 + n_3 + 2$. □

Corollary 3.6. *Suppose condition C_2 holds and $n_1 \le (n_1+2)(n_1-1)/2 \le n_3$, then $\chi_{la}(Ct(3; n_1, 0, n_3)) = n_1 + n_3 + 1$.*

Proof. Under the assumption, $n_1 \ge 2$. Define $f : E(Ct(3; n_1, 0, n_3)) \to [1, n_1 + n_3 + 2]$ such that $f(xy) = 1$, $f(xx_i) = i + 1$ for $1 \le i \le n_1$, $f(yz) = n_1 + 2$ and $f(zz_j) = n_1 + 2 + j$ for $1 \le j \le n_3$. Now, $f^+(y) = n_1 + 3 < f^+(x) = \frac{1}{2}(n_1 + 1)(n_1 + 2)$. Also $f^+(z) > f^+(y)$. Since $(n_1 + 2)(n_1 - 1)/2 \le n_3$ is equivalent to $\frac{1}{2}(n_1 + 1)(n_1 + 2) \le n_1 + n_3 + 2$, we have $f^+(x) = f^+(z_j)$ for some $j \ge 2$. Clearly, f is a local antimagic labeling with $n_1 + n_2 + 1$ distinct vertex colors. By Theorem 2.2 we have $\chi_{la}(Ct(3; n_1, 0, n_3)) = n_1 + n_3 + 1$. □

Problem 3.2. *Study $\chi_{la}(Ct(k; n_1, \dots, n_k))$.*

Note that the authors in [1] have also obtained results on $\chi_{la}(G \odot O_n)$ where G is a path P_m ($m \ge 2$) or a complete graph K_m ($m \ge 4$). Note that $P_m \odot O_n \cong Ct(m; n^{[m]})$. In particular, they showed that $\chi_{la}(Ct(m; n^{[m]})) = |V(Ct(m; n^{[m]}))| = mn + m$ for $m \ge 3, n \ge 2$, and that $\chi_{la}(C_3 \odot O_n) = |V(C_3 \odot O_n)| = 3n + 3$ for $n \ge 2$.

Up to now, it is known that for a graph G of order n, $\chi_{la}(G) = n$ if $G = K_n$ ($n \ge 2$), $K_{1,n-1}$ ($n \ge 3$), or $K(m; n_1, n_2, \dots, n_m)$ for $(n_m+m-1)(n_m+m)/2 >$

$n_1 + n_2 + \cdots + n_m + \binom{m}{2}$, or $Ct(3; n_1, n_2, n_3)$ of Theorem 3.3, or $Ct(m; n^{[m]})$ ($m \geq 3, n \geq 2$) or $C_3 \odot O_n$ ($n \geq 2$). It is also easy to verify that if G is a graph of order $3 \leq n \leq 6$, then $\chi_{la}(G) = n$ if and only if $G = K_n, K_{1,n-1}$ or $K(2; 2, 2)$. In [2, Problem 3.1], the authors posed the problem: Characterize the class of graph G of order n for which $\chi_{la}(G) = n$. We end this paper with the following conjecture.

Conjecture 3.1. *A graph G of order n has $\chi_{la}(G) = n$ if and only if $G = K_n$ ($n \geq 3$); or $K_{1,n-1}$ ($n \geq 3$); or $K(m; n_1, n_2, \ldots, n_m)$ for $(n_m + m - 1)(n_m + m)/2 > n_1 + n_2 + \cdots + n_m + \binom{m}{2}$; or $Ct(3; n_1, n_2, n_3)$ of Theorem 3.3; or $Ct(m; n^{[m]})$ ($m \geq 3, n \geq 2$); or $C_3 \odot O_n$ ($n \geq 2$).*

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