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New Subclass of Close-to-convex Functions Associated with the Vertical Strip Domains

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Abstract. In the present paper, we introduce the class $\mathcal{B}_\theta(\alpha,\beta)$ consisting of functions f, analytic in the unit disc Δ, normalized by the condition $f(0) = 0 = f'(0) - 1$ and satisfying the following two–sided inequality

$$\alpha < \mathrm{Re}\left\{f'(z) + \frac{1+e^{i\theta}}{2} zf''(z)\right\} < \beta \quad (z \in \Delta),$$

where $\theta \in (-\pi, \pi]$, $0 \le \alpha < 1$ and $\beta > 1$. Integral representation, differential subordination results, coefficient estimates and Fekete–Szegö coefficient functional associated with the k–th $(k \ge 1)$ root transform $[f(z^k)]^{1/k}$ for functions in the class $\mathcal{B}_\theta(\alpha,\beta)$, are determined.

Keywords: Univalent, Close–to–convex, Differential subordination, Coefficient estimates, Fekete-Szegö problem.

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1. Introduction

We denote by $\mathcal{H}$ the class of functions f which are holomorphic in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the class of functions $f \in \mathcal{H}$ of the

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form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1.1}$$

which are analytic in the open unit disk Δ and normalized by the condition $f(0) = f'(0) - 1 = 0$. The most familiar subclass of $\mathcal{A}$ consists of univalent functions f in Δ and it is denoted by $\mathcal{S}$. For a univalent function f of the form (1.1), the k–th ($k \geq 1$) root transform is defined by

$$F(z) = [f(z^k)]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1} \quad (z \in \Delta). \tag{1.2}$$

Another well–known classes of analytic functions are

$$\mathcal{P} = \{p \in \mathcal{H} : \; p(0) = 1, \quad \mathrm{Re}\{p(z)\} > 0 \quad \text{in} \quad \Delta\}$$

and

$$\mathcal{B} = \{w \in \mathcal{H} : \; w(0) = 0, \quad |w(z)| < 1 \quad \text{in} \quad \Delta\}.$$

Let f and g be two analytic functions in the class $\mathcal{A}$. We say that f is subordinate to g, written $f(z) \prec g(z)$ or $f \prec g$, if there exists a function $w \in \mathcal{B}$ such that $f(z) = g(w(z))$ for all $z \in \Delta$. Furthermore, if the function g belongs to the class $\mathcal{S}$, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

For a function f given by (1.1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \Delta),$$

their Hadamard product (or convolution), denoted by $f * g$, is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z) \quad (z \in \Delta).$$

Let $\mathcal{S}^*$ and $\mathcal{K}$ be the families of the starlike and convex univalent functions in the open unit disc Δ, respectively. A function $f \in \mathcal{A}$ is called close–to–convex, if there exist $g \in \mathcal{K}$ and $\beta \in \mathbb{R}$ such that

$$\mathrm{Re}\left\{e^{i\beta} \frac{f'(z)}{g'(z)}\right\} > 0, \quad (z \in \Delta).$$

The functions class which satisfy the last condition was introduced by Kaplan in 1952 [5] and we denote it by $\mathcal{CK}$. It is clear that if we let $g(z) \equiv z$ in the class $\mathcal{CK}$, then $\mathcal{CK}$ becomes the Noshiro–Warschawski class

$$\mathcal{C} := \{f \in \mathcal{A} : \exists \beta \in \mathbb{R} : \; \mathrm{Re}\{e^{i\beta} f'(z)\} > 0, \quad z \in \Delta\}.$$

By the basic Noshiro–Warschawski lemma [13, 20], also [2, §2.6], we have $\mathcal{C} \subset \mathcal{S}$.

We recall that a function $f \in \mathcal{A}$ belongs to class $\mathcal{R}$ if it satisfies

$$\mathrm{Re}\{f'(z) + z f''(z)\} > 0 \quad (z \in \Delta).$$

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The class $\mathcal{R}$ was introduced by Chichra [3] and he showed that $\mathcal{R} \subset \mathcal{C}$. Also this class $\mathcal{R}$ was investigated by Singh and Singh [19]. For $0 \le \alpha < 1$ we consider the class $\mathcal{R}_\alpha$ as follows

$$\mathcal{R}_\alpha = \{f \in \mathcal{A} : \quad \mathrm{Re}\,\{f'(z) + zf''(z)\} > \alpha, \quad z \in \Delta\},$$

which was considered by Silverman [17]. In [18], Silverman and Silvia considered the following classes of functions:

$$\mathfrak{L}_\theta := \left\{ f \in \mathcal{A} : \quad \mathrm{Re}\left\{ f'(z) + \frac{1+e^{i\theta}}{2} zf''(z) \right\} > 0, \quad z \in \Delta \right\}$$

and

$$\mathfrak{L}_\theta(b) := \left\{ f \in \mathcal{A} : \quad \left| f'(z) + \frac{1+e^{i\theta}}{2} zf''(z) - b \right| < b, \quad z \in \Delta \right\},$$

where $-\pi < \theta \le \pi$ and $b > 1/2$. Clearly, if $b \to \infty$, then $\mathfrak{L}_\theta(b) \to \mathfrak{L}_\theta$. Very recently Mahzoon and Kargar [12] generalized the class $\mathfrak{L}_\theta$ as follows

$$\mathcal{R}(\theta, \alpha) := \left\{ f \in \mathcal{A} : \quad \mathrm{Re}\left\{ f'(z) + \frac{1+e^{i\theta}}{2} zf''(z) \right\} > \alpha, \quad z \in \Delta \right\},$$

where $0 \le \alpha < 1$. They showed that if $f \in \mathcal{R}(\theta, \alpha)$, then $\mathrm{Re}\{f'(z)\} > \alpha$ and $\mathrm{Re}\{f(z)/z\} > \alpha$ for all $z \in \Delta$. Also, they showed that the class $\mathcal{R}(\theta, \alpha)$ is a subclass of the class of close-to-convex functions.

In recent years, many researchers have introduced new subclass of analytic functions connected with a vertical strip on the complex plane. For instance, Kuroki and Owa [10] introduced the class $\mathcal{S}(\alpha, \beta)$, $(\alpha < 1 < \beta)$, which consists of all functions $f \in \mathcal{A}$ satisfying the following two–sided inequality

$$\alpha < \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < \beta, \quad z \in \Delta.$$

In 2016, Kargar *et al.* [7] introduced the class $\mathcal{V}(\alpha, \beta)$, $0 \le \alpha < 1 < \beta$, as follows:

$$\mathcal{V}(\alpha, \beta) := \left\{ f \in \mathcal{A} : \; \alpha < \mathrm{Re}\left\{ \left(\frac{z}{f(z)} \right)^2 f'(z) \right\} < \beta, \quad z \in \Delta \right\}.$$

In [8] they also introduced the class $\mathcal{M}(\delta)$, $\pi/2 \le \delta < \pi$, including of all functions $f \in \mathcal{A}$ so that

$$1 + \frac{\delta - \pi}{2\sin\delta} < \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\delta}{2\sin\delta}, \quad (z \in \Delta).$$

In this work, motivated by the above definitions, we introduce a new subclass of analytic functions associated with the bounded positive real part. In fact, our class is a certain subclass of the close-to-convex functions.

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Definition 1.1. Let $\theta \in (-\pi, \pi]$, $\alpha \in [0, 1)$ and $\beta \in (1, \infty)$. We say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{B}_\theta(\alpha, \beta)$, if it satisfies the following two-sided inequality

$$\alpha < \mathrm{Re}\left\{f'(z) + \frac{1+e^{i\theta}}{2} z f''(z)\right\} < \beta, \quad (z \in \Delta). \tag{1.3}$$

From among the some choices of θ and α and β which would provide the following known subclasses:

(1) If we let $\beta \to \infty$, then $\mathcal{B}_\theta(\alpha, \beta) \equiv \mathcal{R}(\theta, \alpha)$.
(2) If we let $\theta = 0$ and $\beta \to \infty$, then $\mathcal{B}_\theta(\alpha, \beta) \equiv \mathcal{R}_\alpha$.
(3) If we let $\alpha = 0$ and $\beta \to \infty$, then $\mathcal{B}_\theta(\alpha, \beta) \equiv \mathcal{L}_\theta$.
(4) If we let $\alpha = \theta = 0$ and $\beta \to \infty$, then $\mathcal{B}_\theta(\alpha, \beta) \equiv \mathcal{R}$.
(5) If we let $\alpha = 0$, $\theta = \pi$ and $\beta \to \infty$, then $\mathcal{B}_\theta(\alpha, \beta) \equiv \mathcal{C}$.

In order to prove our results, we need the following lemmas.

Lemma 1.2. (Hallenbeck and Ruscheweyh [4]) *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}$ with* $\mathrm{Re}\{\gamma\} \geq 0$. *If the function $p(z)$ given by $p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in Δ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z) \quad (z \in \Delta), \tag{1.4}$$

then

$$p(z) \prec q(z) \prec h(z) \quad (z \in \Delta), \tag{1.5}$$

where

$$q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(\xi) \xi^{\gamma/n - 1} d\xi.$$

The function q in (1.5) *is the best dominant for p, which satisfies* (1.4)*, in the sense that if $p(z) \prec q_0(z)$, then $q(z) \prec q_0(z)$.*

Lemma 1.3. (Rogosinski [16]) *Let $q(z) = \sum_{n=1}^{\infty} C_n z^n$ be analytic and univalent in Δ and maps Δ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in Δ and satisfies the following subordination*

$$p(z) \prec q(z) \quad (z \in \Delta),$$

then

$$|A_n| \leq |C_1| \quad (n \geq 1).$$

Lemma 1.4. (Keogh and Merkes [9]) *Let the function $g(z)$ given by*

$$g(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

be in the class $\mathcal{P}$. Then, for any complex number μ

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}.$$

The result is sharp for the function $g(z)$ given by

$$g(z) = \frac{1+z}{1-z} \qquad or \qquad g(z) = \frac{1+z^2}{1-z^2}.$$

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This paper is organized as follows. In Section 2 we derive an integral representation and some differential subordination results for functions in the class $\mathcal{B}_\theta(\alpha,\beta)$. In Section 3 coefficient estimates and Fekete-Szegö coefficient functional associated with the k–th root transform $[f(z^k)]^{1/k}$ for function f belonging to the class $\mathcal{B}_\theta(\alpha,\beta)$ are investigated.

2. Integral representation and subordination results

We begin this section with the following lemma. Indeed, it gives a necessary and sufficient condition for functions belonging to the class $\mathcal{B}_\theta(\alpha,\beta)$.

Lemma 2.1. *Let $f\in\mathcal{A}$, $0\le\alpha<1$, $\beta>1$ and $\theta\in(-\pi,\pi]$. Then $f\in\mathcal{B}_\theta(\alpha,\beta)$ if and only if*

$$f'(z)+\frac{1+e^{i\theta}}{2}zf''(z)\prec P_{\alpha,\beta}(z)\quad (z\in\Delta), \tag{2.1}$$

where

$$P_{\alpha,\beta}(z):=1+\frac{\beta-\alpha}{\pi}i\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right). \tag{2.2}$$

Proof. At first, we recall that the function $P_{\alpha,\beta}(z)$ defined by (2.2) is convex univalent in Δ and maps Δ onto the convex domain

$$\Omega_{\alpha,\beta}=\{w\in\mathbb{C}:\alpha<\mathrm{Re}\{w\}<\beta\},$$

conformally (see [10]). By (1.3), $\{f'(z)+[(1+e^{i\theta})/2]zf''(z)\}$ lies in the vertical strip $\Omega_{\alpha,\beta}$ and it is known that $P_{\alpha,\beta}(\Delta)=\Omega_{\alpha,\beta}$. Because $P_{\alpha,\beta}(z)$ is univalent then by the subordination principle, we get (2.1). □

Remark 2.2. The function $P_{\alpha,\beta}(z)$ has the following form

$$P_{\alpha,\beta}(z)=1+\sum_{n=1}^{\infty}B_nz^n, \tag{2.3}$$

where

$$B_n=\frac{\beta-\alpha}{n\pi}i\left(1-e^{2n\pi i\frac{1-\alpha}{\beta-\alpha}}\right)\quad (n=1,2,\ldots). \tag{2.4}$$

Lemma 2.3. *Let $P_{\alpha,\beta}$ be defined as (2.2). Then the function*

$$\lambda(z)=\int_0^1 P_{\alpha,\beta}(zt^{1/\gamma})dt\quad (\mathrm{Re}\{\gamma\}>0, z\in\Delta), \tag{2.5}$$

is convex in Δ.

Proof. Define

$$\psi_\gamma(z):=\int_0^1\frac{1}{1-zt^{1/\gamma}}dt=\sum_{n=0}^{\infty}\frac{\gamma}{n+\gamma}z^n. \tag{2.6}$$

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The function $\psi_\gamma(z)$ is convex in Δ when $\mathrm{Re}\{\gamma\} > 0$ (see [14]). From (2.6) we obtain

$$\begin{aligned} P_{\alpha,\beta}(z) * \psi_\gamma(z) &= P_{\alpha,\beta}(z) * \int_0^1 \frac{1}{1 - zt^{1/\gamma}} dt \\ &= \int_0^1 P_{\alpha,\beta}(zt^{1/\gamma}) dt =: \lambda(z). \end{aligned}$$

On the other hand, since $P_{\alpha,\beta}$ and ψ_γ are convex univalent functions, by the Pòlya–Schoenberg conjecture (this conjecture states that the class of convex univalent functions is preserved under the convolution) that is proved by Ruscheweyh and Sheil–Small (see [15]), the function $\lambda(z)$ is convex univalent in the open unit disk Δ. This completes the proof. □

In the sequel, by the Lemma 2.1, we obtain an integral representation for functions belonging to the class $\mathcal{B}(\alpha, \beta)$.

Theorem 2.4. *Let $\theta \in (-\pi, \pi]$, $\alpha \in [0, 1)$ and $\beta \in (1, \infty)$. Then $f \in \mathcal{B}_\theta(\alpha, \beta)$ if, and only if*

$$f(z) = \int_0^z \frac{\gamma}{\eta^\gamma} \left(\int_0^\eta \xi^{\gamma-1} P_{\alpha,\beta}(w(\xi)) d\xi \right) d\eta \quad (z \in \Delta, \gamma = 2/(1 + e^{i\theta})), \tag{2.7}$$

where $P_{\alpha,\beta}(z)$ is defined by (2.2).

Proof. Let $f \in \mathcal{B}_\theta(\alpha, \beta)$. By definition of subordination and by the Lemma 2.1 there exists a function $w \in \mathcal{B}$ such that

$$f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) = P_{\alpha,\beta}(w(z)) \quad (z \in \Delta). \tag{2.8}$$

From the following identity

$$f'(z) + \frac{1 + e^{i\theta}}{2} z f''(z) = \frac{1 - e^{i\theta}}{2} f'(z) + \frac{1 + e^{i\theta}}{2} (z f'(z))' \quad (z \in \Delta),$$

we get that (2.8) is equivalent to

$$\left(\frac{1 - e^{i\theta}}{1 + e^{i\theta}} \right) f'(z) + (z f'(z))' = \frac{2}{1 + e^{i\theta}} P_{\alpha,\beta}(w(z)) \quad (z \in \Delta). \tag{2.9}$$

If we let

$$\gamma := \frac{2}{1 + e^{i\theta}} \quad (-\pi < \theta < \pi), \tag{2.10}$$

then the above relation (2.9) implies that

$$(\gamma - 1) z^{\gamma-1} f'(z) + z^{\gamma-1} (z f'(z))' = \gamma z^{\gamma-1} P_{\alpha,\beta}(w(z)).$$

Therefore, we find that

$$\left[z^{\gamma-1} \left(z f'(z) \right) \right]' = \gamma z^{\gamma-1} P_{\alpha,\beta}(w(z)),$$

which readily yields

$$z^\gamma f'(z) = \gamma \int_0^z \xi^{\gamma-1} P_{\alpha,\beta}(w(\xi)) d\xi. \tag{2.11}$$

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Integrating once more the equality (2.11), we get (2.7) and concluding the proof. □

Let $t \in [0,1]$ and $\phi \in [0,2\pi)$. By use of the Theorem 2.4, the function

$$f(z,\phi,t) = \int_0^z \frac{\gamma}{\eta^\gamma} \left(\int_0^\eta \xi^{\gamma-1} P_{\alpha,\beta}\left(\frac{e^{i\phi}\xi(\xi+t)}{1+\xi t} \right) d\xi \right) dt \quad (z \in \Delta), \tag{2.12}$$

belongs to the class $\mathcal{B}_\theta(\alpha,\beta)$.

Theorem 2.5. *Let $\theta \in (-\pi,\pi]$, $\alpha \in [0,1)$ and $\beta \in (1,\infty)$. If $f \in \mathcal{B}_\theta(\alpha,\beta)$, then*

$$f'(z) \prec \int_0^1 P_{\alpha,\beta}(zt^{1/\gamma})dt \prec P_{\alpha,\beta}(z) \quad (z \in \Delta, \gamma = 2/(1+e^{i\theta})), \tag{2.13}$$

where $P_{\alpha,\beta}(z)$ is defined by (2.2)*. The function in* (2.13) *is the best dominant for f'.*

Proof. Since $f \in \mathcal{B}_\theta(\alpha,\beta)$, from Lemma 2.1 it follows that (2.1) holds true. If we take $p(z) = f'(z)$, then

$$p(z) + \frac{1}{\gamma} zp'(z) = f'(z) + \frac{1+e^{i\theta}}{2} zf''(z) \prec P_{\alpha,\beta}(z) \quad (z \in \Delta),$$

where γ is defined in (2.10). Therefore, applying the Lemma 1.2 and since $P_{\alpha,\beta}$ is convex univalent function in the open unit disk Δ, we get the best dominant

$$p(z) \prec \frac{\gamma}{z^\gamma} \int_0^z \xi^{\gamma-1} P_{\alpha,\beta}(\xi)d\xi \prec P_{\alpha,\beta}(z) \quad (z \in \Delta). \tag{2.14}$$

Now letting $\xi = zt^{1/\gamma}$ in the above integral (2.14) the differential chain (2.14) implies that (2.13) holds true for all $z \in \Delta$. □

The next theorem is the following.

Theorem 2.6. *Let $\theta \in (-\pi,\pi]$, $\alpha \in [0,1)$ and $\beta \in (1,\infty)$. If $f \in \mathcal{B}_\theta(\alpha,\beta)$, then*

$$\frac{f(z)}{z} \prec \int_0^1 \int_0^1 P_{\alpha,\beta}(zrt^{1/\gamma})drdt \quad (z \in \Delta), \tag{2.15}$$

where $P_{\alpha,\beta}$ is defined by (2.2)*. The function in* (2.15) *is the best dominant for f/z.*

Proof. Let that $p(z) = f(z)/z$. Then by use of (2.13) and applying the Lemma 2.3, and with a simple calculation we have

$$p(z) + zp'(z) = f'(z) \prec \int_0^1 P_{\alpha,\beta}(zt^{1/\gamma})dt \quad (z \in \Delta).$$

If we apply Lemma 1.2 once more with $\gamma = 1$, we conclude that

$$p(z) \prec \frac{1}{z} \int_0^1 \lambda(\xi)d\xi \prec \lambda(z) \quad (z \in \Delta), \tag{2.16}$$

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where λ is defined in (2.5). Now, it is sufficient to put $\xi = rz$ in the integral in (2.16). In this case, if we take into account (2.5), then the first differential subordination in (2.16) implies that (2.15) holds true and concluding the proof. Lemma 1.2 shows that The function in (2.15) is the best dominant for f/z. □

3. On coefficients

We begin this section by estimating the coefficients of members of the family $\mathcal{B}_\theta(\alpha,\beta)$.

Theorem 3.1. *Let $\theta \in (-\pi,\pi]$, $\alpha \in [0,1)$ and $\beta \in (1,\infty)$. Also, let f be of the form* (1.1) *and let it belong to the class $\mathcal{B}_\theta(\alpha,\beta)$. Then*

$$|a_n| \le \frac{2\sqrt{2}(\beta-\alpha)}{n\pi\sqrt{n^2+1+(n^2-1)\cos\theta}}\sin\frac{\pi(1-\alpha)}{\beta-\alpha} \quad (n \ge 2). \tag{3.1}$$

Proof. Let $f \in \mathcal{B}_\theta(\alpha,\beta)$ be of the form (1.1). Then by Lemma (2.1), we have

$$f'(z) + \frac{1+e^{i\theta}}{2}zf''(z) \prec P_{\alpha,\beta}(z) \quad (z \in \Delta). \tag{3.2}$$

If we replace the Taylor-Maclaurin series of f', zf'' and $P_{\alpha,\beta}$ in the above subordination relation (3.2), then we have

$$1 + \sum_{n=1}^{\infty} p_n z^n \prec 1 + \sum_{n=1}^{\infty} B_n z^n.$$

From Lemma 1.3, the last differential subordination implies that

$$|p_n| \le |B_1| \quad (n \ge 1).$$

On the other hand, by equating the coefficients of z^n on both sides of (3.2), the following relation between the coefficients holds true

$$\frac{n}{2}\left[2+(n-1)(1+e^{i\theta})\right]a_n = p_{n-1}. \tag{3.3}$$

Thus, from (3.3), we obtain

$$|a_n| \le \frac{|B_1|}{\frac{n}{2}\left[2+(n-1)(1+e^{i\theta})\right]} = \frac{\sqrt{2}|B_1|}{n\sqrt{n^2+1+(n^2-1)\cos\theta}},$$

where

$$|B_1| = \frac{2(\beta-\alpha)}{\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}.$$

This completes the proof. □

With suitable choices of α, β and θ, we get the following interesting results. If we let $\alpha = 1/2$, $\beta = 3/2$ and $\theta = 0$ in the Theorem 3.1, then we get.

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Corollary 3.2. *If $f \in \mathcal{A}$ of the form* (1.1) *satisfying the following two–sided inequality*

$$\frac{1}{2} < \mathrm{Re}\,\{f'(z) + zf''(z)\} < \frac{3}{2} \quad (z \in \Delta),$$

then

$$|a_n| \leq \frac{2}{\pi n^2} \quad (n \geq 2).$$

If we let $\theta = 0 = \alpha$ and $\beta = 2$ in the Theorem 3.1, then we get.

Corollary 3.3. *If $f \in \mathcal{A}$ of the form* (1.1) *satisfying the following two–sided inequality*

$$0 < \mathrm{Re}\,\{f'(z) + zf''(z)\} < 2 \quad (z \in \Delta),$$

then

$$|a_n| \leq \frac{4}{\pi n^2} \quad (n \geq 2).$$

Recently, many researchers (see e.g. [1], [6], [11]) have considered the Fekete-Szegö functional associated with the k–th root transform for several subclasses of analytic functions. In the next result, we consider this problem for functions in the class $\mathcal{B}_\theta(\alpha, \beta)$. At first, for f given by (1.1), we have

$$(f(z^k))^{1/k} = z + \frac{1}{k}a_2 z^{k+1} + \left(\frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2\right) z^{2k+1} + \cdots . \tag{3.4}$$

Equating the coefficients of (1.2) and (3.4) yields

$$b_{k+1} = \frac{1}{k}a_2 \quad \text{and} \quad b_{2k+1} = \frac{1}{k}a_3 - \frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{3.5}$$

Theorem 3.4. *Let $\theta \in (-\pi, \pi]$, $\alpha \in [0, 1)$ and $\beta \in (1, \infty)$. Also let $f \in \mathcal{B}_\theta(\alpha, \beta)$ and F be the k–th root transform of f defined by* (1.2)*. Then, for any complex number μ, we have*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{2(\beta - \alpha)}{3k\pi\sqrt{5 + 4\cos\theta}} \sin\frac{\pi(1-\alpha)}{\beta - \alpha} \tag{3.6}$$

$$\times \max\left\{1, \left|\frac{1+\zeta}{2} - (2\mu + k - 1)\frac{3(2 + e^{i\theta})(\beta - \alpha)i}{2k\pi(3 + e^{i\theta})^2}(1 - \zeta)\right|\right\},$$

where b_{k+1} and b_{2k+1} are defined in (3.5) *and*

$$\zeta := \exp\left\{\frac{2\pi i(1-\alpha)}{\beta - \alpha}\right\}. \tag{3.7}$$

Proof. If $f \in \mathcal{B}_\theta(\alpha, \beta)$, then by Lemma 2.1 and definition of subordination there exists a function $w \in \mathcal{B}$ such that

$$f'(z) + \frac{1 + e^{i\theta}}{2} zf''(z) = P_{\alpha,\beta}(w(z)). \tag{3.8}$$

Define

$$\mathfrak{p}(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots . \tag{3.9}$$

Since $w \in \mathcal{B}$, it follows that $\mathfrak{p} \in \mathcal{P}$. From (3.9) and (2.3) we have

$$P_{\alpha,\beta}(w(z)) = 1 + \frac{1}{2}B_1p_1z + \left(\frac{1}{4}B_2p_1^2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right)\right)z^2 + \cdots. \quad (3.10)$$

Equating the coefficients of z and z^2 on both sides of (3.8), we get

$$\left(3 + e^{i\theta}\right)a_2 = \frac{1}{2}B_1p_1, \quad (3.11)$$

and

$$\left(6 + 3e^{i\theta}\right)a_3 = \left(\frac{1}{4}B_2p_1^2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right)\right). \quad (3.12)$$

Substituting (3.11) and (3.12) into (3.5), we obtain

$$b_{k+1} = \frac{B_1p_1}{2k\left(3 + e^{i\theta}\right)}$$

and

$$b_{2k+1} = \frac{1}{k\left(6 + 3e^{i\theta}\right)}\left(\frac{1}{4}B_2p_1^2 + \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right)\right) - \frac{(k-1)B_1^2p_1^2}{8k^2\left(3 + e^{i\theta}\right)^2}.$$

Therefore

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{6k\left(2 + e^{i\theta}\right)}\left[p_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} + (2\mu + k - 1)\frac{\left(6 + 3e^{i\theta}\right)B_1}{2k\left(3 + e^{i\theta}\right)^2}\right)p_1^2\right]. \quad (3.13)$$

From (2.4), we have

$$B_1 = \frac{\beta - \alpha}{\pi}i\left(1 - e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right), \quad (3.14)$$

and

$$B_2 = \frac{\beta - \alpha}{2\pi}i\left(1 - e^{4\pi i\frac{1-\alpha}{\beta-\alpha}}\right). \quad (3.15)$$

Putting (3.14) and (3.15) into (3.13) and letting

$$\mu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + (2\mu + k - 1)\frac{\left(6 + 3e^{i\theta}\right)B_1}{2k\left(3 + e^{i\theta}\right)^2}\right),$$

the inequality (3.6) now follows as an application of Lemma 1.4. It is easy to check that the result is sharp for the k–th root transforms of the functions $f(z, \phi, 1)$ where $f(z, \phi, 0)$ is defined in (2.12). □

The problem of finding sharp upper bounds for the coefficient functional $|a_3 - \mu a_2^2|$ for different subclasses of the normalized analytic function class $\mathcal{A}$ is known as the Fekete–Szegö problem. If we take $k = 1$ in Theorem 3.4, we have.

Corollary 3.5. *Let $\theta \in (-\pi, \pi]$, $\alpha \in [0, 1)$ and $\beta \in (1, \infty)$. If $f \in \mathcal{B}_\theta(\alpha, \beta)$, then for any complex number μ we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{2(\beta - \alpha)}{3\pi\sqrt{5 + 4\cos\theta}}\sin\frac{\pi(1 - \alpha)}{\beta - \alpha}$$

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$$\times \max\left\{1, \left|\frac{1+\zeta}{2} - \mu\frac{3(2+e^{i\theta})(\beta-\alpha)i}{2\pi(3+e^{i\theta})^2}(1-\zeta)\right|\right\},$$

where ζ *is given by* (3.7)*. The result is sharp.*

Corollary 3.6. *Let* $0 \le \alpha < 1$ *and* $\beta > 1$*. If* $f \in \mathcal{B}(\alpha,\beta)$*, then for any complex number* μ *we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{2(\beta-\alpha)}{9\pi}\sin\frac{\pi(1-\alpha)}{\beta-\alpha}$$
$$\times \max\left\{1, \left|\frac{1+\zeta}{2} - \mu\frac{9(\beta-\alpha)i}{32\pi}(1-\zeta)\right|\right\}.$$

where ζ *is given by* (3.7)*. The result is sharp.*

If we let $\alpha = 1/2$ and $\beta = 3/2$ in the above Corollary 3.6, we get.

Corollary 3.7. *If* $f \in \mathcal{A}$ *of the form* (1.1) *satisfying the following two–sided inequality*

$$\frac{1}{2} < \mathrm{Re}\{f'(z) + zf''(z)\} < \frac{3}{2} \quad (z \in \Delta),$$

then

$$|a_3 - \mu a_2^2| \le \frac{2}{9\pi}\max\left\{1, \frac{9|\mu|}{8\pi}\right\} \quad (\mu \in \mathbb{C}).$$

If we let $\alpha = 0$ and $\beta = 2$ in Corollary 3.6, we get.

Corollary 3.8. *If* $f \in \mathcal{A}$ *of the form* (1.1) *satisfying the following two–sided inequality*

$$0 < \mathrm{Re}\{f'(z) + zf''(z)\} < 2 \quad (z \in \Delta),$$

then

$$|a_3 - \mu a_2^2| \le \frac{4}{9\pi}\max\left\{1, \frac{9|\mu|}{4\pi}\right\} \quad (\mu \in \mathbb{C}).$$

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