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Common Fixed Points Via R-Functions and Digraphs with an Application to Homotopy Theory

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Abstract. The basic purpose of this article is to introduce the concept of $R-\lambda-G$-contraction by using R-functions, lower semi-continuous functions and digraphs and discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings satisfying such contractions in the setting of metric spaces endowed with a graph. As some consequences of our results, we obtain several recent results in metric spaces and partial metric spaces.

Keywords: R-function, Digraph, Lower semi-continuous function, Common fixed point.

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1. Introduction

Fixed point theory is a branch of nonlinear analysis which can be applied successfully to different fields of mathematics and applied sciences. The Banach contraction principle [3] is a popular tool for solving existence and uniqueness problems in nonlinear analysis. Several researchers successfully extended this interesting result in many directions(see [4, 13, 22, 23]). Recently, a lot of articles have been dedicated to the development of fixed point theory by using

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contractivity conditions that depend on auxiliary functions such as Geraghty functions, altering distance functions, L-functions, simulation functions, manageable functions(see [10, 14, 17, 18, 19, 28, 29]). In 1969, Meir and Keeler [21] introduced a new class of contractive mappings that did not depend on auxiliary functions but have created much attention to a large number of mathematicians. Lim [19] proved that Meir-Keeler contractions and L-functions are closely related to each other. Very recently, Roldán López de Hierro and Shahzad [28] introduced the concept of R-contraction by using the notion of R-function and studied some fixed point results satisfying such contractions. After that, Nastasi et al.[24] introduced a new class of mappings by using R-functions and lower semi-continuous functions and proved some fixed point results in metric spaces and partial metric spaces.

In [16], Jungck introduced the concept of weak compatibility. Several authors have obtained common fixed points by using this notion. In recent investigations, the study of fixed point theory with a graph takes a prominent place in many aspects. In 2005, Echenique [11] studied fixed point theory by using graphs and then Espinola and Kirk [12] applied fixed point results in graph theory. Motivated by the ideas given in [24, 28] and some recent work on metric spaces with a graph (see [2, 5, 6, 6, 7]), we like to introduce the concept of $R - \lambda - G$-contraction that includes Meir-Keeler contractions, Geraghty contractions, $R-\lambda$-contractions, etc. and obtain sufficient conditions for existence and uniqueness of points of coincidence and common fixed points for a pair of mappings in metric spaces endowed with a digraph G. As some consequences of this study, we obtain several recent results in metric spaces and partial metric spaces. Some examples are provided to justify the validity of our results. Finally, we give an application of our main result to homotopy theory.

2. Some Basic Concepts

In this section we present some basic notations, definitions and necessary results in metric spaces.

Definition 2.1. [28] Let $A \subseteq \mathbb{R}$ be a nonempty subset and let $\varrho : A \times A \to \mathbb{R}$ be a function. We say that ϱ is an R-function if it satisfies the following two conditions.

(ϱ_1) If $(a_n) \subset (0, \infty) \cap A$ is a sequence such that $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

(ϱ_2) If (a_n), $(b_n) \subset (0, \infty) \cap A$ are two sequences converging to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then

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$L = 0$.

We denote by R_A the family of all R-functions whose domain is $A \times A$.

In some cases, given a function $\varrho : A \times A \to \mathbb{R}$, we will consider the following property.

(ϱ_3) If (a_n), $(b_n) \subset (0, \infty) \cap A$ are two sequences such that $b_n \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$, then $a_n \to 0$.

Example 2.2. [28] Given $k \in (0, 1)$, let $\varrho : [0, 1] \times [0, 1] \to \mathbb{R}$ be the function given by $\varrho(s, t) = ks - t$ for all t, $s \in [0, 1]$. Then ϱ is an R-function, but it is not a simulation function neither a manageable function because its domain is neither $[0, \infty) \times [0, \infty)$ nor $\mathbb{R} \times \mathbb{R}$.

Example 2.3. [28] Let $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be the function defined, for all t, $s \in [0, \infty)$, by

$$\varrho(t, s) = \begin{cases} \frac{1}{2}s - t, & if\ t < s, \\ \\ 0, & if\ t \geq s. \end{cases}$$

Then ϱ is an R-function on $[0, \infty)$ which also satisfies condition (ϱ_3).

Example 2.4. [31] The following items are some examples of R-functions each of which satisfies condition (ϱ_3).

(a) $\varrho(t, s) = s\phi(s) - t$, where $\phi : [0, \infty) \to [0, 1)$ is a mapping such that $\limsup_{t \to r^+} \varphi(t) < 1$, for all $r \in (0, \infty)$.
(b) $\varrho(t, s) = \frac{s}{t+1} - t$.
(c) $\varrho(t, s) = \frac{s}{e^t} - t$.

Proposition 2.5. [28] *If $\varrho \in R_A$, then $\varrho(a, a) \leq 0$ for all $a \in (0, \infty) \cap A$.*

Proposition 2.6. [28] *If $\varrho(t, s) \leq s - t$ for all t, $s \in A \cap (0, \infty)$, then (ϱ_3) holds.*

Definition 2.7. [14] A Geraghty function is a function $\phi : [0, \infty) \to [0, 1)$ such that if $(t_n) \subset [0, \infty)$ and $\phi(t_n) \to 1$, then $t_n \to 0$.

Definition 2.8. [14] A Geraghty contraction is a mapping $T : (X, d) \to (X, d)$ such that

$$d(Tx, Ty) \leq \phi(d(x, y))d(x, y)\ for\ all\ x, y \in X,$$

where ϕ is a Geraghty function.

Definition 2.9. [21] A Meir-Keeler contraction is a mapping $T : X \to X$ from a metric space (X, d) into itself such that for all $\epsilon > 0$, there exists $\delta > 0$ verifying that if x, $y \in X$ and $\epsilon \leq d(x, y) < \epsilon + \delta$, then $d(Tx, Ty) < \epsilon$.

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Lim characterized this kind of mappings by using the notion of the following class of auxiliary functions.

Definition 2.10. [19] A function $\phi : [0,\infty) \to [0,\infty)$ will be called an L-function if

(a) $\phi(0) = 0$,
(b) $\phi(t) > 0$ for all $t > 0$, and
(c) for all $\epsilon > 0$, there exists $\delta > 0$ such that $\phi(t) \leq \epsilon$ for all $t \in [\epsilon, \epsilon + \delta]$.

Theorem 2.11. [19] *Let (X, d) be a metric space and let $T : X \to X$ be a self mapping. Then T is a Meir-Keeler mapping if and only if there exists an (nondecreasing, right-continuous) L-map ϕ such that*

$$d(Tx, Ty) < \phi(d(x, y)) \ for \ all \ x,\ y \in X \ verifying \ d(x, y) > 0.$$

Theorem 2.12. [28] *Given an L-function $\phi : [0,\infty) \to [0,\infty)$, let $\varrho_\phi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by*

$$\varrho_\phi(t, s) = \phi(s) - t \ for \ all \ t,\ s \in [0,\infty).$$

Then ϱ_ϕ is an R-function on $[0,\infty)$. Furthermore, ϱ_ϕ satisfies condition (ϱ_3).

Lemma 2.13. [28] *If $\phi : [0,\infty) \to [0, 1)$ is a Geraghty function, then $\varrho'_\phi : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined by*

$$\varrho'_\phi(t, s) = \phi(s)s - t \ for \ all \ t,\ s \in [0,\infty),$$

is an R-function on $[0,\infty)$ satisfying condition (ϱ_3).

Let (X, d) be a metric space and $\Re$ be a binary relation over X. Denote $S = \Re \cup \Re^{-1}$. Then

$$x, y \in X,\ xSy \Leftrightarrow x\Re y \ or \ y\Re x.$$

Definition 2.14. [30] We say that (X, d, S) is regular if the following condition holds:

If the sequence (x_n) in X and the point $x \in X$ are such that x_nSx_{n+1} for all $n \geq 1$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then there exists a subsequence (x_{n_i}) of (x_n) such that $x_{n_i}Sx$ for all $i \geq 1$.

Let (X, d) be a metric space. We denote the range of d by

$$ran(d) = \{d(x, y) : x,\ y \in X\} \subset [0,\infty).$$

Definition 2.15. [28] Let (X, d) be a metric space and let $T : X \to X$ be a mapping. We will say that T is an R-contraction if there exists an R-function $\varrho : A \times A \to \mathbb{R}$ such that $ran(d) \subset A$ and

$$\varrho(d(Tx, Ty), d(x, y)) > 0 \ for \ all \ x,\ y \in X \ such \ that \ x \neq y.$$

In such a case, we will say that T is an R-contraction with respect to(w.r.t.) ϱ.

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A real valued function f defined on a metric space (X, d) is said to be lower semi-continuous at a point x_0 in X if $\liminf\limits_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leq \liminf\limits_{x_n \to x_0} f(x_n)$, whenever $x_n \in X$ for each $n \in \mathbb{N}$ and $x_n \to x_0$. We denote by Λ the family of all lower semi-continuous functions $\lambda : (X, d) \to [0, \infty)$. If $\lambda \in \Lambda$, then we will use the following notation

$$D(u, v; \lambda) := d(u, v) + \lambda(u) + \lambda(v) \ for\ all\ u,\, v \in X.$$

Definition 2.16. [24] Let (X, d) be a metric space and let $T : X \to X$ be a mapping. The mapping T is called an $R - \lambda$-contraction w.r.t. ϱ if there exist an R-function $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and a function $\lambda \in \Lambda$ such that

$$\varrho(D(Tx, Ty; \lambda), D(x, y; \lambda)) > 0 \ for\ all\ x,\ y \in X\ with\ D(x, y; \lambda) > 0.$$

Definition 2.17. [1] Let T and S be self mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a coincidence point of T and S and y is called a point of coincidence of T and S.

Definition 2.18. [16] The mappings $T, S : X \to X$ are weakly compatible, if for every $x \in X$, the following holds:

$$T(Sx) = S(Tx)\ whenever\ Sx = Tx.$$

Proposition 2.19. [1] *Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

We next review some basic notions in graph theory.

Let (X, d) be a metric space and G a directed graph such that the vertex set $V(G) = X$ and the set $E(G)$ of its edges contains all the loops, i.e., $\Delta \subseteq E(G)$ where $\Delta = \{(x, x) : x \in X\}$. We also assume that $E(G)$ contains no parallel edges. So we can identify G with the pair $(V(G), E(G))$. We denote the conversion of a graph G by G^{-1}, that is, the graph obtained from G by reversing the direction of the edges i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$. Let $\tilde{G}$ denote the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\tilde{G}$ as a digraph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

Our graph theory notations and terminology are standard and can be found in all graph theory books, like [8, 9, 15]. If x, y are vertices of the digraph G, the direction of edge (x, y) is the inverse of the direction of edge (y, x), that is, $(x, y) \neq (y, x)$. A path of length n $(n \in \mathbb{N})$ in G from x to y is a sequence $(x_i)_{i=0}^{n}$ of $n+1$ distinct vertices such that $x_0 = x,\ x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected.

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Definition 2.20. Let (X, d) be a metric space endowed with a graph $G = (V(G), E(G))$. A mapping $f : X \to X$ is called an $R-\lambda-G$-contraction w.r.t. ϱ if there exist an R-function $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and a function $\lambda \in \Lambda$ such that

$$\varrho(D(fx, fy; \lambda), D(x, y; \lambda)) > 0$$

for all $x, y \in X$ with $(x, y) \in E(G)$ and $D(x, y; \lambda) > 0$.

Definition 2.21. Let (X, d) be a metric space and let $f, g : X \to X$ be mappings. Then f is called an $R-\lambda$-contraction w.r.t. g and ϱ if there exist an R-function $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and a function $\lambda \in \Lambda$ such that

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) > 0$$

for all $x, y \in X$ with $D(gx, gy; \lambda) > 0$.

Definition 2.22. Let (X, d) be a metric space endowed with a graph $G = (V(G), E(G))$ and let $f, g : X \to X$ be mappings. Then f is called an $R-\lambda-G$-contraction w.r.t. g and ϱ if there exist an R-function $\varrho : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and a function $\lambda \in \Lambda$ such that

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) > 0$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$ and $D(gx, gy; \lambda) > 0$.

Remark 2.23. If f is an $R-\lambda-G$-contraction w.r.t. g and ϱ, then f is also an $R-\lambda-G^{-1}$-contraction w.r.t. g and ϱ and hence an $R-\lambda-\tilde{G}$-contraction w.r.t. g and ϱ.

Remark 2.24. If f is an $R-\lambda$-contraction w.r.t. g and ϱ, then f is also an $R-\lambda-G_0$-contraction w.r.t. g and ϱ, where G_0 is the complete graph $(X, X \times X)$. But if f is an $R-\lambda-G$-contraction w.r.t. g and ϱ, it may not be an $R-\lambda$-contraction w.r.t. ϱ (see Remark 3.15).

Definition 2.25. Let (X, d) be a metric space and let $f, g : X \to X$ be mappings. Then f is called continuous w.r.t. g if given $x \in X$ and a sequence $(gx_n)_{n\in\mathbb{N}}$,

$$gx_n \to gx \textit{ implies } fx_n \to fx.$$

Definition 2.26. Let (X, d) be a metric space endowed with a graph $G = (V(G), E(G))$. A mapping $f : X \to X$ is called G-continuous if given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$,

$$x_n \to x \textit{ and } (x_n, x_{n+1}) \in E(G) \textit{ for } n \in \mathbb{N} \textit{ imply } fx_n \to fx.$$

Definition 2.27. Let (X, d) be a metric space endowed with a graph $G = (V(G), E(G))$ and let $f, g : X \to X$ be mappings. Then f is called G-continuous w.r.t. g if given $x \in X$ and a sequence $(gx_n)_{n\in\mathbb{N}}$,

$$gx_n \to gx \textit{ and } (gx_n, gx_{n+1}) \in E(G) \textit{ for } n \in \mathbb{N} \textit{ imply } fx_n \to fx.$$

3. Main Results

In this section we assume that (X, d) is a metric space and G is a reflexive digraph such that $V(G) = X$ and G has no parallel edges. Let the mappings $f, g : X \to X$ be such that $f(X) \subseteq g(X)$. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq g(X)$, there exists an element $x_1 \in X$ such that $fx_0 = gx_1$. Continuing in this way, we can construct a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$.

Definition 3.1. Let (X, d) be a metric space endowed with a graph G and $f, g : X \to X$ be such that $f(X) \subseteq g(X)$. We define C_{gf} the set of all elements x_0 of X such that $(gx_n, gx_m) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$ and for every sequence (gx_n) such that $gx_n = fx_{n-1}$.

Taking $g = I$, the identity map on X, C_{gf} becomes C_f which is the collection of all elements x of X such that $(f^n x, f^m x) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.

Before presenting our main result, we state a property of the graph G, call it property $(*)$.

Property $(*)$**:**
If (gx_n) is a sequence in X such that $gx_n \to x$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Taking $g = I$, the above property reduces to property $(\acute{*})$ which may be stated as follows:

Property $(\acute{*})$**:** If (x_n) is a sequence in X such that $x_n \to x$ and $(x_n, x_{n+1}) \in E(\tilde{G})$ for all $n \geq 1$, then there exists a subsequence (x_{n_i}) of (x_n) such that $(x_{n_i}, x) \in E(\tilde{G})$ for all $i \geq 1$.

Theorem 3.2. *Let (X, d) be a metric space endowed with a graph G and let $f, g : X \to X$ be mappings. Suppose f is an $R - \lambda - G$-contraction w.r.t. g and ϱ, $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of X. Assume that at least one of the following conditions holds:*

(i) *f is $\tilde{G}$-continuous w.r.t. g.*
(ii) *The graph G has the property $(*)$ and the R-function ϱ satisfies condition (ϱ_3).*
(iii) *The graph G has the property $(*)$ and $\varrho(t, s) \leq s - t$ for all t, $s \in (0, \infty)$.*

If $C_{gf} \neq \emptyset$, then f and g have a point of coincidence u in X such that $\lambda(u) = 0$ and for any choice of the starting point $x_0 \in C_{gf}$, the sequence (gx_n) defined by $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u. Moreover, f and g have

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a unique point of coincidence in X if the graph G has the following property:

(∗∗) *If x, y are points of coincidence of f and g in X, then $(x, y) \in E(\tilde{G})$.*

Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Suppose that $C_{gf} \neq \emptyset$. We choose an $x_0 \in C_{gf}$ and keep it fixed. Since $f(X) \subseteq g(X)$, there exists a sequence (gx_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$ and $(gx_n, gx_m) \in E(\tilde{G})$ for m, $n = 0, 1, 2, \cdots$.

If $gx_{j+1} = gx_j$ for some $j \in \mathbb{N} \cup \{0\}$, then $fx_j = gx_{j+1} = gx_j$ which implies that gx_j is a point of coincidence of f and g. We now show that $\lambda(gx_j) = 0$. First note that $gx_{j+1} = gx_j$ implies $gx_i = gx_j$ for all $i \in \mathbb{N} \cup \{0\}$ with $i \geq j$. As g is one to one, $gx_{j+1} = gx_j$ implies $x_{j+1} = x_j$ and so $gx_{j+2} = fx_{j+1} = fx_j = gx_{j+1} = gx_j$. Thus, in general, $gx_i = gx_j$ for all $i \in \mathbb{N} \cup \{0\}$ with $i \geq j$. If possible, suppose $\lambda(gx_j) > 0$. Let $t_i := D(gx_{j+i}, gx_{j+i+1}; \lambda)$ for all $i \in \mathbb{N}$. Then $t_i > 0$ and (gx_{j+i}, gx_{j+i+1}) i.e., $(gx_j, gx_j) \in E(\tilde{G})$. Taking into account that f is an $R - \lambda - G$-contraction w.r.t. g and ϱ, we have

$$\varrho(t_{i+1}, t_i) = \varrho(D(fx_{j+i}, fx_{j+i+1}; \lambda), D(gx_{j+i}, gx_{j+i+1}; \lambda)) > 0$$

for all $i \in \mathbb{N}$. By (ϱ_1), it follows that $t_i \to 0$ as $i \to \infty$ and hence $\lambda(gx_j) = \lambda(gx_{j+i}) \to 0$ as $i \to \infty$. This gives that $\lambda(gx_j) = 0$, which is a contradiction. Therefore, $\lambda(gx_j) = 0$ if $gx_{j+1} = gx_j$ for some $j \in \mathbb{N} \cup \{0\}$.

We now assume that $gx_n \neq gx_{n-1}$ for every $n \in \mathbb{N}$.
We first show that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = 0 \textit{ and } \lim_{n\to\infty} \lambda(gx_n) = 0. \tag{3.1}$$

Let us put $t_n := D(gx_{n-1}, gx_n; \lambda)$ for all $n \in \mathbb{N}$. Then the sequence $(t_n) \subset (0, \infty)$. As f is an $R - \lambda - G$-contraction w.r.t. g and ϱ, we get

$$\begin{aligned} \varrho(t_{n+1}, t_n) &= \varrho(D(gx_n, gx_{n+1}; \lambda), D(gx_{n-1}, gx_n; \lambda)) \\ &= \varrho(D(fx_{n-1}, fx_n; \lambda), D(gx_{n-1}, gx_n; \lambda)) \\ &> 0 \; for \; all \; n \in \mathbb{N}. \end{aligned}$$

By using (ϱ_1), it follows that $t_n \to 0$ as $n \to \infty$. Therefore, $d(gx_{n-1}, gx_n) \to 0$ and $\lambda(gx_n) \to 0$ as $n \to \infty$. Consequently, condition (3.1) holds.

We now show that the sequence (gx_n) is Cauchy in $g(X)$.
Suppose (gx_n) is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two

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subsequences (gx_{n_k}) and (gx_{m_k}) of (gx_n) with $k \leq n_k < m_k$ and

$$d(gx_{n_k}, gx_{m_k-1}) \leq \epsilon < d(gx_{n_k}, gx_{m_k}) \; for \; all \; k \in \mathbb{N}.$$

So, it must be the case that

$$\begin{aligned} \epsilon < d(gx_{n_k}, gx_{m_k}) &\leq d(gx_{n_k}, gx_{m_k-1}) + d(gx_{m_k-1}, gx_{m_k}) \\ &\leq \epsilon + d(gx_{m_k-1}, gx_{m_k}). \end{aligned}$$

As $\lim\limits_{n\to\infty} d(gx_{n-1}, gx_n) = 0$, taking limit as $k \to \infty$, it follows from above that

$$\lim_{k\to\infty} d(gx_{n_k}, gx_{m_k}) = \lim_{k\to\infty} d(gx_{n_k-1}, gx_{m_k-1}) = \epsilon.$$

Taking into account $\lambda(gx_n) \to 0$ as $n \to \infty$, we have

$$\lim_{k\to\infty} D(gx_{n_k}, gx_{m_k}; \lambda) = \lim_{k\to\infty} D(gx_{n_k-1}, gx_{m_k-1}; \lambda) = \epsilon.$$

This allows us to assume that $D(gx_{n_k-1}, gx_{m_k-1}; \lambda) > 0$ for each $k \in \mathbb{N}$. Further note that $(gx_{n_k-1}, gx_{m_k-1}) \in E(\tilde{G})$ for all $k \in \mathbb{N}$. Let us consider two sequences (t_k) and (s_k) such that

$$t_k := D(gx_{n_k}, gx_{m_k}; \lambda) \; and \; s_k := D(gx_{n_k-1}, gx_{m_k-1}; \lambda) \; for \; all \; k \in \mathbb{N}.$$

Then, $(t_k),\ (s_k) \subset (0, \infty)$ and $\lim\limits_{k\to\infty} t_k = \lim\limits_{k\to\infty} s_k = \epsilon$. Since f is an $R-\lambda-G$-contraction w.r.t. g and ϱ, we get

$$\begin{aligned} \varrho(t_k, s_k) &= \varrho(D(gx_{n_k}, gx_{m_k}; \lambda), D(gx_{n_k-1}, gx_{m_k-1}; \lambda)) \\ &= \varrho(D(fx_{n_k-1}, fx_{m_k-1}; \lambda), D(gx_{n_k-1}, gx_{m_k-1}; \lambda)) \\ &> 0 \; for \; all \; k \in \mathbb{N}. \end{aligned}$$

Moreover, $\epsilon < d(gx_{n_k}, gx_{m_k}) \leq D(gx_{n_k}, gx_{m_k}; \lambda) = t_k$ for all $k \in \mathbb{N}$. Thus, condition (ϱ_2) guarantees that $\epsilon = 0$, which is a contradiction. Consequently, it follows that (gx_n) is a Cauchy sequence in $g(X)$. As $g(X)$ is complete, there exists $u \in g(X)$ such that $gx_n \to u = gv$ for some $v \in X$. The lower semi-continuity of λ and condition (3.1) imply that

$$0 \leq \lambda(u) \leq \liminf_{n\to\infty} \lambda(gx_n) = 0.$$

This gives that, $\lambda(u) = 0$.

We now show that u is a point of coincidence of f and g in X. Let us allow to consider the following three cases.

Case-I: Assume that hypothesis (i) holds, i.e., f is $\tilde{G}$-continuous w.r.t. g. In this case, $gx_n \to gv \Rightarrow fx_n \to fv$, that is, $gx_{n+1} \to fv$ and so, $gv = fv = u$. This shows that u is a point of coincidence of f and g.

Case-II: Assume that hypothesis (ii) holds, i.e., the graph G has the property $(*)$ and the R-function ϱ satisfies condition (ϱ_3).

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If there exists a subsequence (gx_{n_i}) of (gx_n) such that $fx_{n_i} = fv$ for all $i \in \mathbb{N}$, then $gx_{n_i+1} = fv$ for all $i \in \mathbb{N}$ and hence $gv = fv = u$. Thus, u is a point of coincidence of f and g in X.

If this does not happen, then we can assume that $x_n \neq v$ and $fx_n \neq fv$ for all $n \in \mathbb{N}$. Since g is one to one, it follows that $gx_n \neq gv$ and $fx_n \neq fv$ for all $n \in \mathbb{N}$. By property $(*)$, there exists a subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, gv) \in E(\tilde{G})$.

Let us consider the sequences (t_i) and (s_i) given by

$$t_i := D(fx_{n_i}, fv; \lambda) \ and \ s_i := D(gx_{n_i}, gv; \lambda) \ for \ all \ i \in \mathbb{N}.$$

Such a choice ensures that (t_i), $(s_i) \subset (0, \infty)$. Obviously, $s_i \to 0$ as $i \to \infty$. Since f is an $R - \lambda - G$-contraction w.r.t. g and ϱ,

$$\varrho(t_i, s_i) = \varrho(D(fx_{n_i}, fv; \lambda), D(gx_{n_i}, gv; \lambda)) > 0$$

for all $i \in \mathbb{N}$. Condition (ϱ_3) guarantees that $t_i \to 0$ as $i \to \infty$. This allows us to obtain that

$$d(gx_{n_i+1}, fv) = d(fx_{n_i}, fv) \to 0 \ as \ i \to \infty$$

which implies that, $d(gv, fv) = 0$ and hence $gv = fv = u$. Thus, u is a point of coincidence of f and g in X.

Case-III: Assume that hypothesis (iii) holds, i.e., the graph G has the property $(*)$ and $\varrho(t, s) \leq s - t$ for all t, $s \in (0, \infty)$.
In this case, Proposition 2.6 ensures that condition (ϱ_3) holds and hence Case-II is also applicable.

Thus, in any case, u is a point of coincidence of f and g in X.

The next is to show that the point of coincidence is unique. Assume that there exists u^* in X such that $fx = gx = u^*$ for some $x \in X$ and $u \neq u^*$. By property $(**)$, we have $(u, u^*) \in E(\tilde{G})$. As $u \neq u^*$, we have

$$t_n := D(u, u^*; \lambda) > 0 \ for \ all \ n \in \mathbb{N}.$$

Then,

$$\begin{aligned} \varrho(t_{n+1}, t_n) &= \varrho(D(u, u^*; \lambda), D(u, u^*; \lambda)) \\ &= \varrho(D(fv, fx; \lambda), D(gv, gx; \lambda)) \\ &> 0 \ for \ all \ n \in \mathbb{N}. \end{aligned}$$

Using condition (ϱ_1), we obtain $t_n \to 0$ as $n \to \infty$, which gives that $d(u, u^*) = 0$, contradicts the fact that $u \neq u^*$. Therefore, $u = u^*$ and so f and g have a unique point of coincidence in X.

If f and g are weakly compatible, then by Proposition 2.19, f and g have a unique common fixed point in X. □

Corollary 3.3. *Let (X, d) be a complete metric space endowed with a graph G and let $f : X \to X$ be an $R - \lambda - G$-contraction w.r.t. ϱ. Assume that at least one of the following conditions holds:*

(i) *f is $\tilde{G}$-continuous.*
(ii) *The graph G has the property $(\acute{*})$ and the R-function ϱ satisfies condition (ϱ_3).*
(iii) *The graph G has the property $(\acute{*})$ and $\varrho(t, s) \leq s-t$ for all $t,\ s \in (0, \infty)$.*

If $C_f \neq \emptyset$, then f has a fixed point u in X such that $\lambda(u) = 0$ and for any choice of the starting point $x_0 \in C_f$, the sequence (x_n) defined by $x_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u. Moreover, f has a unique fixed point in X if the graph G has the following property:

$(\acute{**})$ *If $x,\ y$ are fixed points of f in X, then $(x, y) \in E(\tilde{G})$.*

Proof. The proof can be obtained from Theorem 3.2 by considering $g = I$, the identity map on X. □

Corollary 3.4. *Let (X, d) be a metric space and let $f,\ g : X \to X$ be mappings. Suppose f is an $R - \lambda$-contraction w.r.t. g and ϱ, $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of X. Assume that at least one of the following conditions holds:*

(i) *f is continuous w.r.t. g.*
(ii) *The R-function ϱ satisfies condition (ϱ_3).*
(iii) *$\varrho(t, s) \leq s - t$ for all $t,\ s \in (0, \infty)$.*

Then f and g have a unique point of coincidence u in X such that $\lambda(u) = 0$ and for any choice of the starting point $x_0 \in X$, the sequence (gx_n) defined by $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. The proof follows from Theorem 3.2 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$.

□

The following corollary is the Theorem 3.1 of Nastasi et al.[24].

Corollary 3.5. [24] *Let (X, d) be a complete metric space and let $f : X \to X$ be an $R - \lambda$-contraction w.r.t. ϱ. Assume that at least one of the following conditions holds:*

(i) *f is continuous.*
(ii) *The R-function ϱ satisfies condition (ϱ_3).*
(iii) *$\varrho(t, s) \leq s - t$ for all $t,\ s \in (0, \infty)$.*

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Then f has a unique fixed point u in X such that $\lambda(u) = 0$ and for any choice of the starting point $x_0 \in X$, the sequence (x_n) defined by $x_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u.

Proof. The proof can be obtained from Theorem 3.2 by considering $g = I$ and $G = G_0$. □

Corollary 3.6. *Let (X, d) be a complete metric space endowed with a binary relation $\Re$ over X and let $S = \Re \cup \Re^{-1}$. Suppose $f : X \to X$ is such that there exist an R-function ϱ on $[0, \infty)$ and a function $\lambda \in \Lambda$ satisfying*

$$\varrho(D(fx, fy; \lambda), D(x, y; \lambda)) > 0$$

for all x, $y \in X$ with xSy and $D(x, y; \lambda) > 0$. Suppose also that the following conditions hold:

(i) *(X, d, S) is regular and the R-function ϱ satisfies condition (ϱ_3);*
(ii) *there exists $x_0 \in X$ such that $(f^n x_0, f^m x_0) \in S$ for all m, $n = 0, 1, 2, \cdots$.*

Then f has a fixed point u in X such that $\lambda(u) = 0$. Moreover, f has a unique fixed point in X if the following property holds:

If x, y are fixed points of f in X, then xSy.

Proof. The proof follows from Theorem 3.2 by taking $g = I$ and $G = (V(G), E(G))$, where $V(G) = X$, $E(G) = \{(x, y) \in X \times X : xSy\} \cup \Delta$. □

Corollary 3.7. *Let (X, d) be a metric space and let f, $g : X \to X$ be mappings. Suppose $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of X and there exists a $\lambda \in \Lambda$ such that*

$$D(fx, fy; \lambda) \leq \phi(D(gx, gy; \lambda))\, D(gx, gy; \lambda) \tag{3.2}$$

for all x, $y \in X$ with $D(gx, gy; \lambda) > 0$, where ϕ is a Geraghty function. Then f and g have a unique point of coincidence u in X such that $\lambda(u) = 0$ and for any choice of the starting point $x_0 \in X$, the sequence (gx_n) defined by $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. We define a function $\varphi : [0, \infty) \to [0, 1)$ by

$$\varphi(t) = \frac{1}{2}(1 + \phi(t)) \; for \; all \; t \in [0, \infty).$$

Then φ is also a Geraghty function. Moreover, it follows that $\phi(t) < \varphi(t) < 1$ for all $t \in [0, \infty)$. Therefore, from condition (3.2), we get

$$D(fx, fy; \lambda) \leq \phi(D(gx, gy; \lambda))\, D(gx, gy; \lambda) < \varphi(D(gx, gy; \lambda))\, D(gx, gy; \lambda) \tag{3.3}$$

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for all $x,\, y \in X$ with $D(gx, gy; \lambda) > 0$, where φ is a Geraghty function. By applying Lemma 2.13, it follows that

$$\varrho(t, s) = \varphi(s)s - t \ for \ all \ s,\, t \in [0, \infty)$$

is an R-function on $[0, \infty)$ which satisfies condition (ϱ_3). By using condition (3.3), we obtain

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) = \varphi(D(gx, gy; \lambda))\, D(gx, gy; \lambda) - D(fx, fy; \lambda) > 0$$

for all $x,\, y \in X$ with $D(gx, gy; \lambda) > 0$. This shows that f is an $R{-}\lambda$-contraction w.r.t. g and ϱ. The result now follows from Corollary 3.4. ☐

Remark 3.8. Geraghty's fixed point theorem [14] can be obtained from Corollary 3.7, by taking $g = I$ and $\lambda(x) = 0$ for all $x \in X$. Therefore, Theorem 3.2 is a generalization of Geraghty's fixed point theorem.

Corollary 3.9. *Let (X, d) be a metric space and let $f,\, g : X \to X$ be mappings. Suppose $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of X and there exists a $\lambda \in \Lambda$ such that*

$$D(fx, fy; \lambda) < \phi(D(gx, gy; \lambda)) \tag{3.4}$$

for all $x,\, y \in X$ with $D(gx, gy; \lambda) > 0$, where ϕ is an L-function. Then f and g have a unique point of coincidence u in X such that $\lambda(u) = 0$ and for any choice of the starting point $x_0 \in X$, the sequence (gx_n) defined by $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. If ϕ is an L-function, then Theorem 2.12 guarantees that $\varrho(t, s) = \phi(s) - t$ for all $t,\, s \in [0, \infty)$, is an R-function on $[0, \infty)$ which satisfies condition (ϱ_3). By using condition (3.4), we obtain

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) = \varphi(D(gx, gy; \lambda)) - D(fx, fy; \lambda) > 0$$

for all $x,\, y \in X$ with $D(gx, gy; \lambda) > 0$. This shows that f is an $R{-}\lambda$-contraction w.r.t. g and ϱ. The result now follows from Corollary 3.4. ☐

Remark 3.10. Meir-Keeler's fixed point theorem [21] can be obtained from Corollary 3.9, by taking $g = I$, $\lambda(x) = 0$ for all $x \in X$ and using Theorem 2.11. Therefore, Theorem 3.2 is a generalization of Meir-Keeler's fixed point theorem.

Corollary 3.11. *Let (X, d) be a metric space and let $f,\, g : X \to X$ be mappings. Suppose $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of X and there exists a $\lambda \in \Lambda$ such that*

$$\phi(D(fx, fy; \lambda)) < D(gx, gy; \lambda) \tag{3.5}$$

for all $x,\, y \in X$ with $D(gx, gy; \lambda) > 0$, where $\phi : [0, \infty) \to [0, \infty)$ is a right continuous function such that $\phi(t) > t$, for all $t > 0$. Then f and g have a

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unique point of coincidence u in X such that $\lambda(u) = 0$ and for any choice of the starting point $x_0 \in X$, the sequence (gx_n) defined by $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. We consider as R-function

$$\varrho(t, s) = s - \phi(t) \; for \; all \; s, t \in [0, \infty)$$

which satisfies condition (ϱ_3). By using condition (3.5), we obtain

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) = D(gx, gy; \lambda) - \varphi(D(fx, fy; \lambda)) > 0$$

for all $x, y \in X$ with $D(gx, gy; \lambda) > 0$. This shows that f is an $R{-}\lambda$-contraction w.r.t. g and ϱ. The result now follows from Corollary 3.4. □

Corollary 3.12. *Let (X, d) be a metric space and let $f, g : X \to X$ be mappings. Suppose $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of X and there exists a $\lambda \in \Lambda$ such that*

$$D(fx, fy; \lambda) \leq \phi(D(gx, gy; \lambda))\, D(gx, gy; \lambda)$$

for all $x, y \in X$ with $D(gx, gy; \lambda) > 0$, where $\phi : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{t \to r^+} \phi(t) < 1$, for all $r > 0$. Then f and g have a unique point of coincidence u in X such that $\lambda(u) = 0$ and for any choice of the starting point $x_0 \in X$, the sequence (gx_n) defined by $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. We define a function $\varphi : [0, \infty) \to [0, 1)$ by

$$\varphi(t) = \frac{1}{2}(1 + \phi(t)) \; for \; all \; t \in [0, \infty).$$

Then φ is a function such that $\limsup_{t \to r^+} \varphi(t) < 1$, for all $r > 0$. Moreover, it follows that $\phi(t) < \varphi(t) < 1$ for all $t \in [0, \infty)$. By an argument similar to that used in Corollary 3.7 and taking

$$\varrho(t, s) = \varphi(s)s - t \; for \; all \; s, t \in [0, \infty)$$

as R-function which satisfies condition (ϱ_3), the desired result follows from Corollary 3.4. □

Corollary 3.13. *Let (X, d) be a metric space and let $f, g : X \to X$ be mappings. Suppose $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of X and there exists a $\lambda \in \Lambda$ such that*

$$D(fx, fy; \lambda) < \frac{D(gx, gy; \lambda)}{1 + D(fx, fy; \lambda)}$$

for all $x, y \in X$ with $D(gx, gy; \lambda) > 0$. Then f and g have a unique point of coincidence u in X such that $\lambda(u) = 0$ and for any choice of the starting point

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$x_0 \in X$, the sequence (gx_n) defined by $gx_n = fx_{n-1}$ for each $n \in \mathbb{N}$ converges to the point u. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Considering

$$\varrho(t,s) = \frac{s}{1+t} - t \text{ for all } s,\, t \in [0,\infty)$$

as R-function on $[0,\infty)$ which satisfies condition (ϱ_3), the desired result can be obtained from Corollary 3.4. □

We furnish some examples in favour of our main result.

EXAMPLE 3.14. Let $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}$ by $d(x,y) = \mid x - y \mid$ for all $x,\, y \in X$. Then (X,d) is a complete metric space. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{n}) : n = 1,\, 2,\, 3,\, \cdots\}$. Let $f,\ g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{5}, & if\ x \neq \frac{2}{5}, \\ \\ 1, & if\ x = \frac{2}{5} \end{cases}$$

and $gx = 8x$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and g is one to one.

Let $\varrho : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ be defined by

$$\varrho(t,s) = \begin{cases} \frac{1}{2}s - t, & if\ t < s, \\ \\ 0, & if\ t \geq s. \end{cases}$$

Then ϱ is an R-function on $[0,\infty)$ which also satisfies (ϱ_3). We define the lower semi-continuous function $\lambda : X \to [0,\infty)$ by $\lambda(x) = \mid x \mid$ for all $x \in X$.

If $x = 0,\ y = \frac{1}{8n},\ n \in \mathbb{N}$, then $gx = 0,\ gy = \frac{1}{n}$ and so $(gx, gy) \in E(G)$ and $D(gx, gy; \lambda) > 0$ with $D(fx, fy; \lambda) < D(gx, gy; \lambda)$. Then, for $x = 0,\ y = \frac{1}{8n}$, we have

$$\begin{aligned} \varrho(D(fx,fy;\lambda), D(gx,gy;\lambda)) &= \frac{1}{2}D(gx,gy;\lambda) - D(fx,fy;\lambda) \\ &= \frac{1}{2}(\mid gx - gy \mid + \mid gx \mid + \mid gy \mid) \\ &\quad -(\mid fx - fy \mid + \mid fx \mid + \mid fy \mid) \\ &= \frac{1}{n} - \frac{1}{20n} \\ &= \frac{19}{20n} \\ &> 0. \end{aligned}$$

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If $x, y \in X$ with $x = y \neq 0$, then $(gx, gy) \in E(G)$, $D(gx, gy; \lambda) > 0$ and $D(fx, fy; \lambda) < D(gx, gy; \lambda)$. Thus, for $x = y \neq 0$, we have

$$\begin{aligned}
\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) &= \frac{1}{2}D(gx, gy; \lambda) - D(fx, fy; \lambda) \\
&= \frac{1}{2}(2 \mid gx \mid) - 2 \mid fx \mid \\
&= (8 \mid x \mid - \frac{2}{5} \mid x \mid) > 0 \; if \; x \neq \frac{2}{5} \\
&= (\frac{16}{5} - 2) > 0 \; if \; x = \frac{2}{5}.
\end{aligned}$$

Therefore,

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) > 0$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$ with $D(gx, gy; \lambda) > 0$ and so, f is an $R - \lambda - G$-contraction w.r.t. g and ϱ.

We now verify that $0 \in C_{gf}$. In fact, $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$ gives that $gx_1 = f0 = 0 \Rightarrow x_1 = 0$ and so $gx_2 = fx_1 = 0 \Rightarrow x_2 = 0$. Proceeding in this way, we get $gx_n = 0$ for $n = 0, 1, 2, \cdots$ and hence $(gx_n, gx_m) = (0, 0) \in E(\tilde{G})$ for $m, n = 0, 1, 2, \cdots$.
Also, any sequence (gx_n) with the property $(gx_n, gx_{n+1}) \in E(\tilde{G})$ must be either a constant sequence or a sequence of the following form

$$gx_n = \begin{cases} 0, & if \; n \; is \; odd, \\ \\ \frac{1}{n}, & if \; n \; is \; even \end{cases}$$

where the words 'odd' and 'even' are interchangeable. Consequently it follows that property $(*)$ holds. Furthermore, f and g are weakly compatible. Thus, we have all the conditions of Theorem 3.2 and 0 is the unique common fixed point of f and g in X such that $\lambda(0) = 0$.

Remark 3.15. In Example 3.14, f is not an $R - \lambda$-contraction w.r.t. ϱ. In fact, for $x = 0$, $y = \frac{2}{5}$, we have $D(fx, fy; \lambda) = \mid fx - fy \mid + \mid fx \mid + \mid fy \mid = 2$, $D(x, y; \lambda) = \frac{4}{5} > 0$. As $D(fx, fy; \lambda) > D(x, y; \lambda)$, it follows that

$$\varrho(D(fx, fy; \lambda), D(x, y; \lambda)) = 0.$$

We now examine the necessity of property $(*)$ in Theorem 3.2.

EXAMPLE 3.16. Let $X = [0, \infty)$ and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = \mid x - y \mid$ for all $x, y \in X$. Then (X, d) is a complete metric space. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(x, y) : (x, y) \in (0, 1] \times (0, 1], x \geq y\}$. Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{6}, & if \; x \neq 0, \\ \\ 1, & if \; x = 0 \end{cases}$$

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and $gx = \frac{x}{2}$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and g is one to one.

Let $\varrho : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ be defined by

$$\varrho(t,s) = \begin{cases} \frac{1}{2}s - t, & if\ t < s, \\ \\ 0, & if\ t \geq s. \end{cases}$$

Then ϱ is an R-function on $[0, \infty)$ which also satisfies (ϱ_3). We define the lower semi-continuous function $\lambda : X \to [0, \infty)$ by $\lambda(x) = x$ for all $x \in X$.

Since for $x, y \in X$ with $(gx, gy) \in E(G)$ and $D(gx, gy; \lambda) > 0$ imply that $x \neq 0,\ y \neq 0,\ D(fx, fy; \lambda) < D(gx, gy; \lambda)$, we have

$$\begin{aligned} \varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) &= \frac{1}{2}D(gx, gy; \lambda) - D(fx, fy; \lambda) \\ &= \frac{1}{2}D(gx, gy; \lambda) - (\mid fx - fy \mid + fx + fy) \\ &= \frac{1}{2}D(gx, gy; \lambda) - (\mid \frac{x}{6} - \frac{y}{6} \mid + \frac{x}{6} + \frac{y}{6}) \\ &= \frac{1}{2}D(gx, gy; \lambda) - \frac{1}{3}D(gx, gy; \lambda) \\ &= \frac{1}{6}D(gx, gy; \lambda) \\ &> 0. \end{aligned}$$

Therefore, f is an $R-\lambda-G$-contraction w.r.t. g and ϱ. Moreover, it is easy to check that $1 \in C_{gf}$ i.e., $C_{gf} \neq \emptyset$.

We see that f and g have no point of coincidence in X. This happens due to lack of property $(*)$. For instance, we consider the sequence (gx_n), where $x_n = \frac{2}{n}$. Then, $gx_n \to 0$ and $(gx_n, gx_{n+1}) \in E(\tilde{G})$ for all $n \in \mathbb{N}$. But there exists no subsequence (gx_{n_i}) of (gx_n) such that $(gx_{n_i}, 0) \in E(\tilde{G})$.

The following two examples show that the uniqueness part of Theorem 3.2 remains invalid without property $(**)$ of the graph G.

Example 3.17. Let $X = [0, \infty)$ and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = \mid x - y \mid$ for all $x,\ y \in X$. Then (X, d) is a complete metric space. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$. Let $f,\ g : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{3}, & if\ 0 \leq x \leq 1, \\ \\ x^2, & if\ x > 1 \end{cases}$$

and $gx = 3x$ for all $x \in X$. Obviously, $f(X) \subseteq g(X) = X$ and g is one to one.

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We take $\varrho : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ defined by

$$\varrho(t, s) = \begin{cases} \frac{1}{2}s - t, & if\ t < s, \\ 0, & if\ t \geq s \end{cases}$$

as an R-function on $[0, \infty)$ which also satisfies (ϱ_3). We consider as lower semi-continuous function $\lambda(x) = 0$ for all $x \in X$.

If $x = 0,\ y = \frac{1}{3n},\ n \in \mathbb{N}$, then $gx = 0,\ gy = \frac{1}{n}$ and so $(gx, gy) \in E(G)$ and $D(gx, gy; \lambda) > 0$ with $D(fx, fy; \lambda) < D(gx, gy; \lambda)$. Then, for $x = 0,\ y = \frac{1}{3n}$, we have

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) = \frac{1}{2}D(gx, gy; \lambda) - D(fx, fy; \lambda) = \frac{1}{2n} - \frac{1}{9n} = \frac{7}{18n} > 0.$$

Therefore,

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) > 0$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$ and $D(gx, gy; \lambda) > 0$ which states that, f is an $R - \lambda - G$-contraction w.r.t. g and ϱ.

Moreover, $0 \in C_{gf} \neq \emptyset$ and property $(*)$ holds. We find that 0 and 9 are points of coincidence of f and g in X but $(0, 9) \notin E(\tilde{G})$. In fact, unique point of coincidence of f and g does not exist due to lack of property $(**)$ of the graph G.

Example 3.18. Let $X = \{1, 2, 3\} \cup [4, \infty)$ and define $d : X \times X \to \mathbb{R}$ by $d(x, y) = \mid x - y \mid$ for all $x, y \in X$. Then (X, d) is a complete metric space. Let G be a digraph such that $V(G) = X$ and $E(G) = \Delta \cup \{(1, 2)\}$. Let $f, g : X \to X$ be defined by

$$fx = \begin{cases} 2, & if\ x = 1, 2, \\ 3, & if\ x = 3, \\ x^2, & if\ x \geq 4 \end{cases}$$

and

$$gx = \begin{cases} x, & if\ x = 1, 2, 3, \\ x + 1, & if\ x \geq 4. \end{cases}$$

Obviously, $f(X) \subseteq g(X)$, $g(X)$ is a complete subspace of X and g is one to one.

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We take $\varrho : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ defined by

$$\varrho(t,s) = \begin{cases} \frac{1}{2}s - t, & if\ t < s, \\ \\ 0, & if\ t \geq s \end{cases}$$

as an R-function on $[0,\infty)$ which also satisfies (ϱ_3). We consider as lower semi-continuous function $\lambda(x) = 0$ for all $x \in X$.

If $x = 1,\ y = 2$, then $gx = 1,\ gy = 2$ and so $(gx, gy) \in E(G)$ and $D(gx, gy; \lambda) > 0$ with $D(fx, fy; \lambda) < D(gx, gy; \lambda)$. Then, for $x = 1,\ y = 2$, we have

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) = \frac{1}{2}D(gx, gy; \lambda) - D(fx, fy; \lambda) = \frac{1}{2} > 0.$$

Therefore,

$$\varrho(D(fx, fy; \lambda), D(gx, gy; \lambda)) > 0$$

for all $x, y \in X$ with $(gx, gy) \in E(G)$ and $D(gx, gy; \lambda) > 0$ which shows that, f is an $R - \lambda - G$-contraction w.r.t. g and ϱ.

It is easy to verify that property $(*)$ holds and $C_{gf} \neq \emptyset$. Thus, we have all the conditions of Theorem 3.2 except property $(**)$. However, f and g are weakly compatible, we can not find unique common fixed point of f and g. In fact, 2 and 3 are common fixed points of f and g in X and hence they are also points of coincidence of f and g in X, but $(2,3) \notin E(\tilde{G})$.

4. Common fixed points in partial metric spaces

In this section we present some common fixed point theorems in partial metric spaces. We begin with some basic definitions and notions in partial metric spaces that can be found in [20, 25, 26, 27].

Definition 4.1. A partial metric on a nonempty set X is a function $p : X \times X \to [0,\infty)$ such that, for all $u,\ v,\ w \in X$, we have

(i) $u = v \Leftrightarrow p(u,u) = p(u,v) = p(v,v)$;
(ii) $p(u,u) \leq p(u,v)$;
(iii) $p(u,v) = p(v,u)$;
(iv) $p(u,v) \leq p(u,w) + p(w,v) - p(w,w)$.

A partial metric space is a pair (X,p), where X is a nonempty set and p is a partial metric on X.

Every partial metric $p : X \times X \to [0,\infty)$ generates a T_0 topology τ_p on X, which has as a base the family of open p-balls $\{U_p(u,r) : u \in X,\ r > 0\}$, where $U_p(u,r) = \{v \in X : p(u,v) < p(u,u) + r\}$ for all $u \in X$ and $r > 0$.

Definition 4.2. Let (X,p) be a partial metric space and $(u_j) \subset X$. Then

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(i) (u_j) converges to a point $u \in X$ if and only if $p(u,u) = \lim_{j\to\infty} p(u,u_j)$;

(ii) (u_j) is called a Cauchy sequence if there exists $\lim_{i,\,j\to\infty} p(u_i,u_j)$ (and it is finite);

(iii) (X,p) is said to be complete if every Cauchy sequence (u_j) in X converges, with respect to τ_p, to a point $u \in X$ such that $p(u,u) = \lim_{i,\,j\to\infty} p(u_i,u_j)$.

It is elementary to verify that the function $d^p : X\times X \to [0,\infty)$ defined by

$$d^p(u,v) = 2p(u,v) - p(u,u) - p(v,v)$$

is a metric on X whenever p is a partial metric on X. Moreover, $\lim_{j\to\infty} d^p(u_j,u) = 0$ if and only if

$$p(u,u) = \lim_{j\to\infty} p(u_j,u) = \lim_{i,\,j\to\infty} p(u_i,u_j).$$

Lemma 4.3. [24] *Let (X,p) be a partial metric space and let $\lambda : X \to [0,\infty)$ be defined by $\lambda(u) = p(u,u)$ for all $u \in X$. Then the function λ is continuous in the metric space (X,d^p).*

Lemma 4.4. [20, 25] *Let (X,p) be a partial metric space. Then*

(i) *(u_j) is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d^p);*

(ii) *a partial metric space (X,p) is complete if and only if the metric space (X,d^p) is complete.*

Theorem 4.5. *Let (X,p) be a partial metric space endowed with a graph G and let $f,\, g : X \to X$ be mappings. Suppose there exists an R-function $\varrho : [0,\infty)\times[0,\infty) \to \mathbb{R}$ such that*

$$\varrho(p(fx,fy),p(gx,gy)) > 0 \tag{4.1}$$

for all $x,\, y \in X$ with $gx \neq gy$ and $(gx,gy) \in E(G)$. Suppose also that $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of (X,p). Assume that at least one of the following conditions holds:

(i) *f is $\tilde{G} - d^p$-continuous w.r.t. g.*

(ii) *The graph G has the property $(*)$ in (X,d^p) and the R-function ϱ satisfies condition (ϱ_3).*

(iii) *The graph G has the property $(*)$ in (X,d^p) and $\varrho(t,s) \leq s - t$ for all $t,\, s \in (0,\infty)$.*

If $C_{gf} \neq \emptyset$, then f and g have a point of coincidence u in X such that $p(u,u) = 0$. Moreover, f and g have a unique point of coincidence in X if the graph G has the property $(**)$. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Since $d^p(u,v) = 2\,p(u,v) - p(u,u) - p(v,v)$, it follows that

$$p(u,v) = \frac{d^p(u,v) + p(u,u) + p(v,v)}{2} \; for\ all\ u,\, v \in X. \tag{4.2}$$

As $g(X)$ is a complete subspace of (X,p), Lemma 4.4 ensures that $g(X)$ is a complete subspace of (X,d) where $d = \frac{1}{2}d^p$. Let the function $\lambda : X \to [0,\infty)$ be defined by $\lambda(u) = \frac{1}{2}p(u,u)$. Then, by Lemma 4.3, λ is continuous and hence lower semi-continuous in (X,d). From condition (4.2), we get

$$p(u,v) = d(u,v) + \lambda(u) + \lambda(v) = D(u,v;\lambda).$$

Thus, condition (4.1) reduces to

$$\varrho(D(fx,fy;\lambda), D(gx,gy;\lambda)) > 0$$

for all $x,\, y \in X$ with $(gx,gy) \in E(G)$ and $D(gx,gy;\lambda) > 0$. This shows that f is an $R{-}\lambda{-}G$-contraction w.r.t. g and ϱ. Consequently, it follows that we have all the conditions of Theorem 3.2 w.r.t. the metric space (X,d). Therefore, the conclusion of Theorem 4.5 follows from Theorem 3.2 where $p(u,u) = 2\,\lambda(u) = 0$. □

Corollary 4.6. *Let (X,p) be a complete partial metric space endowed with a graph G and let $f : X \to X$ be a mapping. Suppose there exists an R-function $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that*

$$\varrho(p(fx,fy), p(x,y)) > 0$$

for all $x,\, y \in X$ with $x \neq y$ and $(x,y) \in E(G)$. Assume that at least one of the following conditions holds:

(i) *f is $\tilde{G} - d^p$-continuous.*

(ii) *The graph G has the property $(*)'$ in (X,d^p) and the R-function ϱ satisfies condition (ϱ_3).*

(iii) *The graph G has the property $(*)'$ in (X,d^p) and $\varrho(t,s) \leq s - t$ for all $t,\, s \in (0,\infty)$.*

If $C_f \neq \emptyset$, then f has a fixed point u in X such that $p(u,u) = 0$. Moreover, f has a unique fixed point in X if the graph G has the property $(**)'$.

Proof. The proof can be obtained from Theorem 4.5 by considering $g = I$, the identity map on X. □

Corollary 4.7. *Let (X,p) be a partial metric space and let $f,\, g : X \to X$ be mappings. Suppose there exists an R-function $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that*

$$\varrho(p(fx,fy), p(gx,gy)) > 0$$

for all $x,\, y \in X$ with $gx \neq gy$. Also, suppose that $f(X) \subseteq g(X)$, g is one to one and $g(X)$ is a complete subspace of (X,p). Assume that at least one of the following conditions holds:

(i) *f is d^p-continuous w.r.t. g.*

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(ii) *The R-function* ϱ *satisfies condition* (ϱ_3).
(iii) $\varrho(t,s) \le s-t$ *for all* $t,\, s \in (0,\infty)$.

Then f *and* g *have a unique point of coincidence* u *in* X *such that* $p(u,u)=0$. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X.

Proof. The proof can be obtained from Theorem 4.5 by considering $G = G_0$. □

Corollary 4.8. [24] *Let* (X,p) *be a complete partial metric space and let* $f : X \to X$ *be a mapping. Suppose that there exists an R-function* $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ *such that*

$$\varrho(p(fx,fy),p(x,y)) > 0$$

for all $x,\, y \in X$ *with* $x \neq y$. *Assume that at least one of the following conditions holds:*

(i) f *is* d^p*-continuous.*
(ii) *The R-function* ϱ *satisfies condition* (ϱ_3).
(iii) $\varrho(t,s) \le s-t$ *for all* $t,\, s \in (0,\infty)$.

Then f *has a unique fixed point* u *in* X *such that* $p(u,u)=0$.

Proof. The proof can be obtained from Theorem 4.5 by considering $g = I$ and $G = G_0$. □

Corollary 4.9. *Let* (X,p) *be a partial metric space and let* $f,\, g : X \to X$ *be mappings. Suppose* $f(X) \subseteq g(X)$, g *is one to one and* $g(X)$ *is a complete subspace of* X *such that*

$$p(fx,fy) \le \phi(p(gx,gy))\, p(gx,gy)$$

for all $x,\, y \in X$ *with* $gx \neq gy$, *where* ϕ *is a Geraghty function. Then* f *and* g *have a unique point of coincidence* u *in* X *such that* $p(u,u)=0$ *and for any choice of the starting point* $x_0 \in X$, *the sequence* (gx_n) *defined by* $gx_n = fx_{n-1}$ *for each* $n \in \mathbb{N}$ *converges to the point* u. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* X.

Proof. By an argument similar to that used in Corollary 3.7, we can obtain the desired result from Theorem 4.5. □

The following is the Matthews fixed point theorem [20].

Corollary 4.10. [20] *Let* (X,p) *be a complete partial metric space and let* $f : X \to X$ *be a mapping. Suppose that there exists* $k \in [0,1)$ *such that*

$$p(fx,fy) \le k\, p(x,y) \tag{4.3}$$

for all $x,\, y \in X$ *with* $x \neq y$. *Then* f *has a unique fixed point* $u \in X$ *such that* $p(u,u)=0$.

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Proof. Given $k \in [0,1)$, we can find $k' \in (0,1)$ such that $k < k'$. Then, we obtain from condition (4.3) that

$$p(fx, fy) \leq k\, p(x,y) < k'\, p(x,y)$$

for all x, $y \in X$ with $x \neq y$. Considering as R-function $\varrho(t,s) = k's - t$ for all t, $s \in [0,\infty)$ with $k' \in (0,1)$, the result follows from Corollary 4.8. □

Remark 4.11. Geraghty type fixed point theorem in partial metric spaces can be obtained from Corollary 4.9, by taking $g = I$. Several existing fixed point results in the setting of partial metric spaces can also be obtained from Theorem 4.5 by considering suitable R-functions.

5. An application

In this section, we present a homotopy result for operators on a nonempty set endowed with a metric and a digraph. Let Γ denote the family of all nondecreasing upper semi-continuous functions $\rho : [0,\infty) \to [0,\infty)$ such that $\rho(s) < s$ for all $s > 0$ with the following property:

$$\lim_{i,j\to\infty} [s_{i,j} - \rho(s_{i,j})] = 0 \textit{ implies } \lim_{i,j\to\infty} s_{i,j} = 0 \tag{5.1}$$

for every sequence $(s_{i,j}) \subset [0,\infty)$.

Lemma 5.1. *If $\rho \in \Gamma$, then $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by*

$$\varrho(t,s) = \rho(s) - t \textit{ for all } t,\, s \in [0,\infty)$$

is an R-function on $[0,\infty)$ satisfying condition (ϱ_3).

Proof. (ϱ_1) Assume that $(a_n) \subset (0,\infty)$ is a sequence such that $\varrho(a_{n+1}, a_n) > 0$ for all $n \in \mathbb{N}$. Then,

$$0 < \varrho(a_{n+1}, a_n) = \rho(a_n) - a_{n+1}.$$

Since $a_n > 0$ and $\rho(s) < s$ for all $s > 0$, we have

$$a_{n+1} < \rho(a_n) < a_n \textit{ for all } n \in \mathbb{N}.$$

Hence, (a_n) is a strictly decreasing sequence of positive real numbers and so it is convergent. Let $L \geq 0$ be its limit. If possible, suppose that $L > 0$. Therefore,

$$0 < L < a_{n+1} < \rho(a_n) < a_n \textit{ for all } n \in \mathbb{N}.$$

Taking limit as $n \to \infty$, we get $\lim\limits_{n\to\infty} \rho(a_n) = L$. As ρ is upper semi-continuous,

$$\rho(L) \geq \limsup_{r\to L} \rho(r) = \lim_{n\to\infty} \rho(a_n) = L.$$

This contradicts the fact that $\rho(s) < s$ for all $s > 0$. Therefore, $L = 0$.

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(ϱ_2) Assume that (a_n), $(b_n) \subset (0, \infty)$ converging to the same limit $L \geq 0$ and verifying that $L < a_n$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$. In order to prove that $L = 0$, we assume that $L > 0$. Therefore,

$$0 < \varrho(a_n, b_n) = \rho(b_n) - a_n \ for \ all \ n \in \mathbb{N}.$$

So, it must be the case that

$$L < a_n < \rho(b_n) < b_n \ for \ all \ n \in \mathbb{N}.$$

Letting $n \to \infty$, we deduce that $\lim_{n \to \infty} \rho(b_n) = L$. As ρ is upper semi-continuous,

$$\rho(L) \geq \limsup_{r \to L} \rho(r) = \lim_{n \to \infty} \rho(b_n) = L.$$

This again contradicts the fact that $\rho(s) < s$ for all $s > 0$. Therefore, $L = 0$.

(ϱ_3) Let (a_n), $(b_n) \subset (0, \infty)$ be two sequences such that $b_n \to 0$ and $\varrho(a_n, b_n) > 0$ for all $n \in \mathbb{N}$. Therefore,

$$0 < \varrho(a_n, b_n) = \rho(b_n) - a_n < b_n - a_n \ for \ all \ n \in \mathbb{N}.$$

So,

$$0 < a_n < b_n \ for \ all \ n \in \mathbb{N},$$

which implies that $a_n \to 0$. □

Theorem 5.2. *Let (X, d) be a complete metric space endowed with a graph G, let U be an open subset of X such that $(x, y) \in E(\tilde{G})$ for all x, $y \in U$ and V be a closed subset of X with $U \subset V$. Suppose the graph G has the property $(\acute{*})$. Assume that the operator $T : V \times [0, 1] \to X$ satisfies the following conditions:*

(i) *$u \neq T(u, s)$ for each $u \in V \setminus U$ and all $s \in [0, 1]$;*

(ii) *there exists $\rho \in \Gamma$ such that for each $s \in [0, 1]$ and all u, $v \in V$ with $(u, v) \in E(G)$, we have*

$$d(T(u, s), T(v, s)) \leq \rho(d(u, v));$$

(iii) *there exists a continuous function $f : [0, 1] \to \mathbb{R}$ such that $d(T(u, t), T(u, s)) \leq \mid f(t) - f(s) \mid$ for all t, $s \in [0, 1]$ and every $u \in V$.*

Then $T(\cdot, 1)$ has a fixed point if $T(\cdot, 0)$ has a fixed point and $C_{T(\cdot, s)} \neq \emptyset$ for every fixed $s \in [0, 1]$ with the property that $T(\cdot, s)$ is a self mapping on a closed subset of U.

Proof. We first note that if $\rho \in \Gamma$, then $\rho_1 : [0, \infty) \to [0, \infty)$ defined by $\rho_1(s) = \frac{1}{2}(s + \rho(s))$ satisfies $\rho(s) < \rho_1(s)$ for all $s > 0$ and $\rho_1 \in \Gamma$.

Suppose that $T(\cdot, 0)$ has a fixed point. We consider the set $S := \{s \in [0, 1] : u = T(u, s) \ for \ some \ u \in U\}$. As $T(\cdot, 0)$ has a fixed point, hypothesis (i) guarantees that there exists $u \in U$ such that $u = T(u, 0)$ and so $0 \in S$. This implies that S is nonempty. We shall show that S is both open and closed in

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$[0,1]$. As $[0,1]$ is connected, it follows that $S=[0,1]$.

We now prove that S is a closed subset of $[0,1]$. Let $(s_j)\subset S$ be a sequence such that $s_j\to s_0\in[0,1]$ as $j\to\infty$. Since $(s_j)\subset S$, for each $j\in\mathbb{N}$, there exists $u_j\in U$ such that $u_j=T(u_j,s_j)$. As $(u_i,u_j)\in E(\tilde{G})$, by using hypotheses (ii) and (iii), we obtain

$$\begin{aligned} d(u_i,u_j) &= d(T(u_i,s_i),T(u_j,s_j)) \\ &\leq d(T(u_i,s_i),T(u_i,s_j))+d(T(u_i,s_j),T(u_j,s_j)) \\ &\leq \mid f(s_i)-f(s_j)\mid +\rho(d(u_i,u_j)) \end{aligned}$$

which implies that

$$d(u_i,u_j)-\rho(d(u_i,u_j))\leq\mid f(s_i)-f(s_j)\mid$$

for all i, $j\in\mathbb{N}$. Taking limit as i, $j\to\infty$, we get

$$\lim_{i,j\to\infty}[d(u_i,u_j)-\rho(d(u_i,u_j))]=0.$$

Condition (5.1) ensures that $d(u_i,u_j)\to 0$ as i, $j\to\infty$ and consequently it follows that (u_j) is a Cauchy sequence. Since (X,d) is complete and V is closed, there exists $u\in V$ such that $u_j\to u$. By property $(*)'$ of the graph G, there exists a subsequence (u_{j_k}) of (u_j) such that $(u_{j_k},u)\in E(\tilde{G})$.

Then, assuming $d(u_{j_k},u)>0$, we get

$$\begin{aligned} d(u_{j_k},T(u,s_0)) &= d(T(u_{j_k},s_{j_k}),T(u,s_0)) \\ &\leq d(T(u_{j_k},s_{j_k}),T(u_{j_k},s_0))+d(T(u_{j_k},s_0),T(u,s_0)) \\ &\leq \mid f(s_{j_k})-f(s_0)\mid +\rho(d(u_{j_k},u)) \\ &< \mid f(s_{j_k})-f(s_0)\mid +d(u_{j_k},u). \end{aligned}$$

If $d(u_{j_k},u)=0$, then $u_{j_k}=u=T(u,s_{j_k})$ and so

$$\begin{aligned} d(u_{j_k},T(u,s_0)) &= d(T(u,s_{j_k}),T(u,s_0)) \\ &\leq \mid f(s_{j_k})-f(s_0)\mid \\ &= \mid f(s_{j_k})-f(s_0)\mid +d(u_{j_k},u). \end{aligned}$$

Thus, in any case, we have

$$d(u_{j_k},T(u,s_0))\leq\mid f(s_{j_k})-f(s_0)\mid +d(u_{j_k},u).$$

Taking limit as $k\to\infty$, we get

$$d(u,T(u,s_0))=\lim_{k\to\infty}d(u_{j_k},T(u,s_0))=0.$$

This shows that $u=T(u,s_0)$ and by hypothesis (i), we get $u\in U$. Therefore, $s_0\in S$ and consequently, it follows that S is closed.

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Finally, we show that S is an open subset of $[0,1]$. Let $s_0 \in S$ be arbitrary. Then there exists $u_0 \in U$ such that $u_0 = T(u_0, s_0)$. As U is open in (X,d), one finds $r > 0$ such that $B[u_0, r] = \{x \in X : d(u_0, x) \leq r\} \subset U$. Using continuity of f at s_0, corresponding to $\epsilon = r - \rho(r) > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\mid f(s) - f(s_0) \mid < \epsilon$ for all $s \in (s_0 - \delta, s_0 + \delta)$. For $s \in (s_0 - \delta, s_0 + \delta)$ and $u \in B[u_0, r]$, we have

$$\begin{aligned} d(T(u,s), u_0) &= d(T(u,s), T(u_0, s_0)) \\ &\leq d(T(u,s), T(u,s_0)) + d(T(u,s_0), T(u_0,s_0)) \\ &\leq \mid f(s) - f(s_0) \mid + \rho(d(u,u_0)) \\ &\leq r - \rho(r) + \rho(d(u,u_0)) \\ &\leq r, \ \textit{as } \rho \textit{ is nondecreasing}. \end{aligned}$$

This shows that $T(\cdot, s)$ is a self mapping on $B[u_0, r]$ for every fixed $s \in (s_0 - \delta, s_0 + \delta)$. Let $\varrho : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by $\varrho(t,s) = \rho_1(s) - t$ for all t, $s \geq 0$. By Lemma 5.1, it follows that ϱ is an R-function on $[0,\infty)$ satisfying condition (ϱ_3). Hypothesis (ii) ensures that $T(\cdot, s)$ is an $R-\lambda-G$-contraction w.r.t. ϱ on the closed ball $B[u_0, r]$, where $\lambda \in \Lambda$ defined by $\lambda(x) = 0$ for all $x \in B[u_0, r]$. Thus, all hypotheses of Corollary 3.3 are satisfied which ensures that $T(\cdot, s)$ has a fixed point in $B[u_0, r]$ and hence in U. Consequently, it follows that $(s_0 - \delta, s_0 + \delta) \subset S$ and so S is an open subset of $[0,1]$. Preceding discussion guarantees that $S = [0,1]$ and so $1 \in S$ which gives that there exists $u^* \in U$ such that $u^* = T(u^*, 1)$ i.e., u^* is a fixed point of $T(\cdot, 1)$. □

Remark 5.3. It is worth mentioning that Theorem 5.1 of Nastasi et al.[24] can be obtained from Theorem 5.2 by considering $G = G_0$.

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