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Some Classes of Weakly Prime Center Rings

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Abstract. Let R be a ring. The ring R is called weakly prime center(WPC ring) if $ab \in Z(R)$ implies that aRb is an ideal of R. In this paper, we prove that every left(right) duo ring is a WPC ring. Also we introduce some classes of rings with nilpotent Jacobson radical which are WPC rings. Finally, we prove that a simple ring is a WPC ring if and only if it is a domain.

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1. Introduction

In this paper, all rings have identity elements. The class of weakly prime center rings or simply WPC rings is introduced in [4]. In [4] some properties of these rings was investigated. Also the relation of WPC rings and other classes of rings such as clean rings, exchange rings and semi periodic rings has been studied. A ring R is called weakly prime center(WPC ring) if $ab \in Z(R)$ implies that aRb is an ideal of R. It is clear that every commutative ring is a WPC ring. So WPC rings are a generalization of commutative rings. In this paper, we prove that every left(right) duo ring is a WPC ring. Also we prove that a simple ring is a WPC ring if and only if it is a domain. Also we correct a mistake in [4].

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Here we give some notions and definitions. We denote the center of a ring R by $Z(R)$. Also the *Jacobson* radical of the ring R is denoted by $J(R)$. A ring with no non-zero nilpotent element is called a *reduced* ring. A ring R is called *reversible* if $ab = 0$ implies that $ba = 0$. For example every reduced ring is reversible. A ring R is called *semicommutative* if $ab = 0$ implies that $aRb = 0$. Every reversible ring is semicommutative. A ring R is called an *abelian ring* if every idempotent element is central. Semicommutative rings are abelian rings. A ring R is called *left(right) duo* ring if every left(right) ideal of R is a two sided ideal. A ring R is called left(right) quasiduo if every maximal left(right) ideal of R is an ideal. A ring R is *directly finite* if $ab = 1$ implies that $ba = 1$ for any $a, b \in R$. An ideal A of a ring R is *idempotent-lifting* if $A \subseteq J(R)$ and every idempotent of $\frac{R}{A}$ has the form $x + A$, where x is an idempotent of R. A ring R is *semi-local* if $\frac{R}{J(R)}$ is a semisimple artinian ring. A semi-local ring R is *semi-perfect* if $J(R)$ is idempotent-lifting. A ring R is a Von Neumann regular ring if $a \in aRa$ for each $a \in R$. A ring R is a strongly regular ring if $a \in a^2R$ for each $a \in R$. It is well known that a ring R is a strongly regular ring if and only if R is a Von Neumann regular ring and R is a reduced ring. For any other unexplained notations or definitions see [2] and [3].

2. Main results

In this section we state our main results.

Lemma 2.1. *Let R be a ring and $a, b \in R$*

(1) *If $ab \in U(R)$, then $aRb = R$.*
(2) *Assume $ab \in Z(R)$. If $a \in U(R)$ or $b \in U(R)$, then $aRb = abR = Rab$.*

Proof. (1) Assume $abu = uab = 1$. Then $aR = R = Rb$. So $aRb = Rb = R$.
(2) Assume $a \in U(R)$. Then $aR = R = Ra$. So $aRb = Rb = Rab = abR$.
□

Lemma 2.2. *If R is a semicommutative ring whose center is a field, then R is a WPC ring.*

Proof. Let $Z(R) = F$. If $ab \in F$, then $ab = 0$ or $ab \in F^* \subseteq U(R)$. If $ab = 0$, then $aRb = 0$ is a two sided ideal. If $ab \in F^* \subseteq U(R)$, then $aRb = R$ by Lemma 2.1. □

Corollary 2.3. *If R is a reduced ring whose center is a field, then R is a WPC ring.*

Example 2.4. If k is a field and $R = k\langle X\rangle$ is the free algebra on set X, then R is a WPC ring.

The following proposition of [4] gives a class of local WPC rings.

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Theorem 2.5. [4, Proposition 2.2] *Let R be a local ring. If $J(R)$ is commutative, then R is WPC.*

Corollary 2.6. *Let R be a local ring. If $J(R)^2 = 0$, then R is a WPC ring.*

EXAMPLE 2.7. If R is a local ring, then $R/J(R)^2$ is a WPC ring by Corollary 2.6.

Theorem 2.8. *Let F be field and $R = F \bigoplus J(R)$ be a local $F-$algebra. If $J(R)^4 = 0$ and $(J(R)^2 \bigcap Z(R))J(R) = 0$, then R is a WPC ring.*

Proof. Let $ab \in Z(R)$. Note that aRb is an $F-$vector space. If $a \in U(R)$ or $b \in U(R)$, then aRb is a two sided ideal by Lemma 2.1. So assume $a, b \in J(R)$. Hence $aJ(R)bJ(R) = 0$. Also $ab \in J(R)^2 \bigcap Z(R)$. So $abJ(R) = 0$. Hence $aRbJ(R) = aFbJ(R) + aJ(R)bJ(R) = abJ(R) + aJ(R)bJ(R) = 0$. Therefore $aRbR = aRbF + aRbJ(R) = aRb + abJ(R) = aRb$. Similarly $RaRb = aRb$. So aRb is an ideal and R is a WPC ring. □

Corollary 2.9. *Let F be a field. If* $R = \{ \begin{bmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & g \\ 0 & 0 & 0 & a \end{bmatrix} : a, b, c, d, e, f, g \in F\}$, *then R is a WPC ring.*

Proof. Let $I = \{ \begin{bmatrix} 0 & b & c & d \\ 0 & 0 & e & f \\ 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 \end{bmatrix} : b, c, d, e, f, g \in F\}$. Then $R/I \cong F$ and $I^4 = 0$. Hence $I = J(R)$ and $R = F \bigoplus J(R)$. Also $Z(R) = \{ \begin{bmatrix} a & 0 & 0 & d \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} : a, d \in F\}$. So $(J(R)^2 \bigcap Z(R))J(R) = 0$. Hence R is a WPC ring by Theorem 2.8. □

Corollary 2.10. *Let F be field. If $R = F \bigoplus J(R)$ is a local $F-$algebra and $J(R)^3 = 0$, then R is a WPC ring.*

Proof. It is clear that $(J(R)^2 \bigcap Z(R))J(R) = 0$. Hence R is a WPC ring by Theorem 2.8. □

Corollary 2.11. *Let F be a field. If* $R = \{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} : a, b, c, d \in F\}$, *then R is a WPC ring.*

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Proof. Since $J(R) = \{\begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} : b, c, d \in F\}$, we conclude that $J(R)^3 = 0$. Hence R is a WPC ring by Corollary 2.10. □

Remark 2.12. Let $R_n = \{\begin{bmatrix} a & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a & a_{2,3} & \cdots & a_{2,n} \\ 0 & 0 & a & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{bmatrix} : a, a_{i,j} \in F\}$. Then R_3 is a semicommutative ring by [1, Proposition 1.2]. But R_n is not semicommutative for $n \geq 4$. Let $A = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Then $AB = 0$ and $ARB = \{\begin{bmatrix} 0 & 0 & 0 & f & f \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} : f \in F\}$. So $C \in ARB$. But $CD = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \notin ARB$. So ARB is not an ideal. Hence R_5 is not a WPC ring. A similar proof shows that R_n is not a WPC ring for $n \geq 5$.

Theorem 2.13. *If R is a left(right) duo ring, then R is a WPC ring.*

Proof. Assume R is a left duo ring. Let $ab \in Z(R)$. Then $aR \subseteq Ra$ and $bR \subseteq Rb$. So $abR \subseteq aRb \subseteq Rab = abR$. Hence $aRb = Rab = abR$ is a two sided ideal. □

Remark 2.14. Since every strongly regular ring is duo ring, strongly regular rings are WPC rings. This is a part of [4, Proposition 2.6].

For the proof of the next theorem we need the following lemma.

Lemma 2.15. [4, Lemma 2.1.] *The WPC rings are directly finite.*

In [4, Remark 2.3] it is proved that for a division ring D, $M_2(D)$ is a simple ring which is not a WPC ring. In the next theorem, we classify all simple WPC rings.

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Theorem 2.16. *A simple ring R is a WPC ring if and only if it is a domain.*

Proof. Since R is a simple ring, $Z(R)$ is a field. If R is a domain, then the proof follows from Corollary 2.3. Conversely, assume R is a simple WPC ring. If $ab = 0$, then aRb is an ideal. So $aRb = 0$ or $aRb = R$. If $aRb = R$, then there exists an $r \in R$ such that $arb = 1$. Since WPC rings are directly finite, $rba = 1$. This implies that $b = rbab = 0$. Hence $R = aRb = 0$ which is a contradiction. So $aRb = 0$. Since $ann_l(Rb)$ is a two sided ideal, $ann_l(Rb) = 0$ or $ann_l(Rb) = R$. If $ann_l(Rb) = 0$, then $a = 0$. If $ann_l(Rb) = R$, then $b = 0$. So R is a domain. □

The following Lemma is an interesting property of idempotents in WPC rings which is stated in the proof of [4, Theorem 3.1-part 1].

Lemma 2.17. *Let R be a WPC ring. If $e \in R$ is an idempotent element, then $eR(1-e)Re = 0$. In particular, $(eR(1-e)R)^2 = 0$ and $(R(1-e)Re)^2 = 0$ and $eR(1-e)R, R(1-e)Re \subseteq J(R)$.*

Proof. The equation $e(1-e) = 0$ implies that $eR(1-e)$ is an ideal of R. Hence $eR(1-e)R \subseteq eR(1-e)$. So $eR(1-e)Re \subseteq eR(1-e)Re \subseteq eR(1-e)e = 0$. Hence $(eR(1-e)R)^2 = (R(1-e)Re)^2 = 0$ which implies that $eR(1-e)R, R(1-e)Re \subseteq J(R)$. □

Theorem 2.18. *Let R be a WPC ring. If $J(R)$ is idempotent-lifting, then $R/J(R)$ is an abelian ring.*

Proof. Let $x \in R/J(R)$ be an idempotent of the ring $R/J(R)$. So there is an idempotent $e \in R$ such that $x = e + J(R)$. So $re - ere = (1-e)re \in J(R)$ and $er(1-e) = er - ere \in J(R)$ by Lemma 2.17. This implies that $re - er \in J(R)$. So $x \in Z(R/J(R))$. Hence $R/J(R)$ is an abelian ring. □

Theorem 2.19. *Let R be a WPC ring. If R is a semi-perfect ring, then $R/J(R)$ is direct product of finitely many division ring. In particular, R is quasiduo ring.*

Proof. Since $R/J(R)$ is an Artinian semisimple ring, it is isomorphic to direct product of finitely many matrix rings over some division rings by Artin-Wedderburn theorem. Also $J(R)$ is an idempotent-lifting ideal. So $R/J(R)$ is an abelian ring by Theorem 2.18. Hence $R/J(R)$ is a product of finitely many division rings. Since $R/J(R)$ is a quasiduo ring, R is a quasiduo ring. □

Corollary 2.20. *If R is an Artinian WPC ring, then R is a quasiduo ring.*

In [4, Remark 2.4] it is mentioned that the ring $R = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z}_5\}$ is not a WPC ring. But its computations are not correct. In fact, we prove that R is a WPC ring.

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Theorem 2.21. *Let F be a field. If $R = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in F\}$, then R is a WPC ring.*

Proof. First note that $Z(R) = \{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in F\}$ is a field. Let $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R$ and $B = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} \in R$ be such that $AB = \begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix} \in Z(R)$. If $0 \neq AB \in Z(R)$, then $AB \in U(R)$ and $ARB = R$ by Lemma 2.1. If $AB = 0$, then $ARB \subseteq \{\begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} : d \in F\} = J(R)$. The set ARB is an $F-$subspace of $J(R)$ and $dim_F J(R) = 1$. So $ARB = 0$ or $ARB = J(R)$. Hence R is a WPC ring.

□

Remark 2.22. The ring $R = \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in F\}$ is not a semicommutative ring since it is not an abelian ring. So R is an example of a WPC ring which is not semicommutative. Hence the converse of Theorem 2.2 is not true.

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